11th AIMS Conf. on Dynamical Systems, DEs & Applications, Honoring Peter Lax's 90th Birthday

Orlando, FL, 5 July 2016

A Mathematical Theory of Climate Sensitivity:

A Tale of Deterministic & Stochastic Dynamical Systems

Michael Ghil

Ecole Normale Supérieure, Paris, and
University of California, Los Angeles

Based on joint work with **A. Bracco, M.D. Chekroun, D. Kondrashov, H. Liu, J.C. McWilliams, J.D. Neelin, S. Pierini, E. Simonnet, S. Wang & I. Zaliapin**

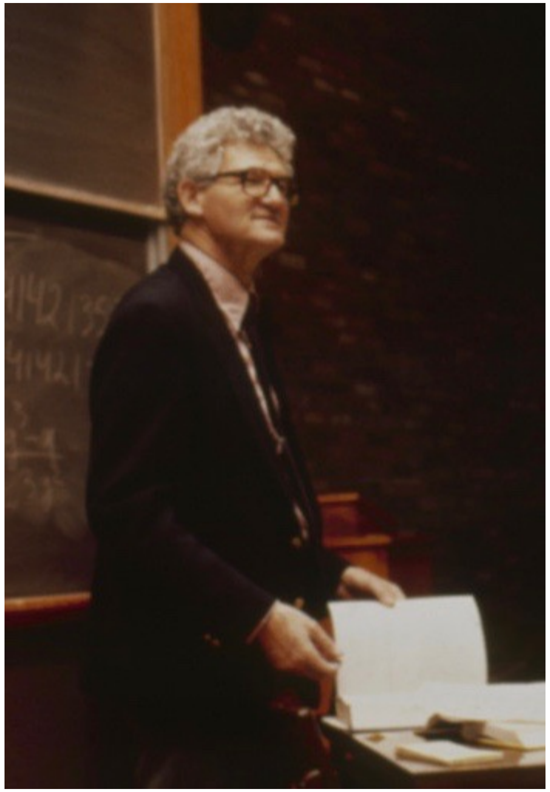
Please visit these sites for more info.

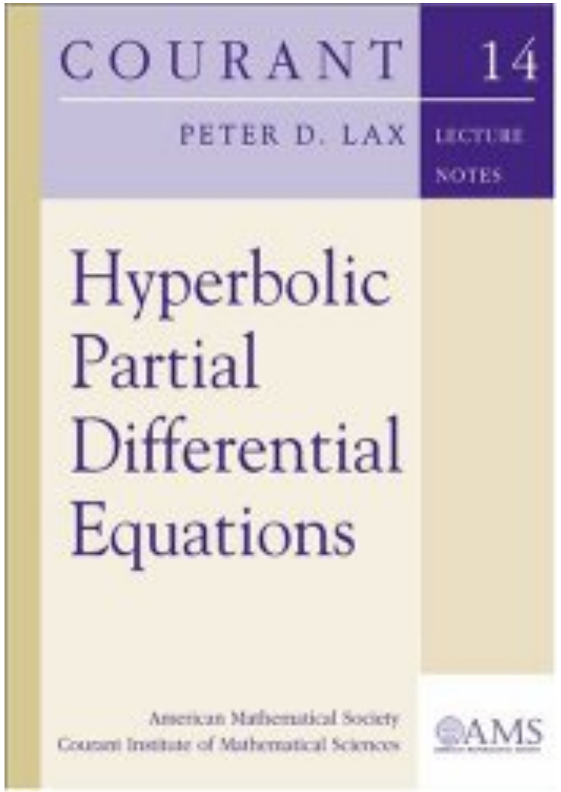
http://www.atmos.ucla.edu/tcd/

http://www.environnement.ens.fr/

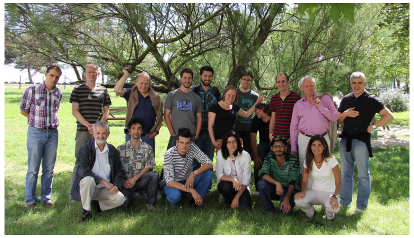
Overall Outline

- A selection of Peter Lax contributions





- Dynamical systems & climate (MG & friends)



P. D. Lax Contributions – Selected!

- The Lax equivalence theorem in numerical analysis
- Specific numerical schemes for hyperbolic PDEs
  - Lax-Friedrichs – first-order accurate, positive
  - Lax-Wendroff – second-order accurate
- Hyperbolic systems of conservation laws

  – the entropy condition
- ∞-dimensional Hamiltonian systems
  - Lax equation and Lax pairs for the KdV equation
  - Generalization to a large class of such PDEs
  - Extensions to localized coherent structures in geophysical flows
- "PDE Lax"

  – Lax-Milgram theorem & its application to the finite-element method

  – Hopf-Lax-Oleinik formula for the Hamilton-Jacobi equation
- The "Lax school"

P. D. Lax Contributions – I, Numerical Analysis

➢ The Lax equivalence theorem in numerical analysis

Theorem. Given the consistency of a finite-difference scheme for an evolution problem (i.e., that it formally solves the hyperbolic or parabolic PDE at hand), convergence ←→ stability (Lax & Richtmyer, *CPAM*, 1956).

Proof. Stability → convergence is pretty easy to prove, while convergence → stability is not, and the latter requires a functional-analysis trick.

Remark. In practice, the useful observation is that, if it's consistent and it doesn't blow up in your face, it will eventually converge, as $\Delta t \to 0$.

This result has been called the **Fundamental Theorem of Numerical Analysis:** "[It] gave us work to do, precise results to prove, something to accomplish with our analysis and our lives."(*)

(*) G. Strang, in his review of the book

Peter Lax, Mathematician: An Illustrated Memoir, 2015,

by Reuben Hersh, American Mathematical Society, Providence, RI;

see *SIAM News*, May 1, 2015.

P. D. Lax Contributions – I, Numerical Analysis

- Practical finite-difference schemes for hyperbolic PDEs

– Lax-Friedrichs scheme

$$u_t + au_x = 0, \quad u_i^n = u(x_i, t_n), \quad x_i = i\Delta x, t_n = n\Delta t;$$

$$u_i^{(n+1)} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - a\frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$$

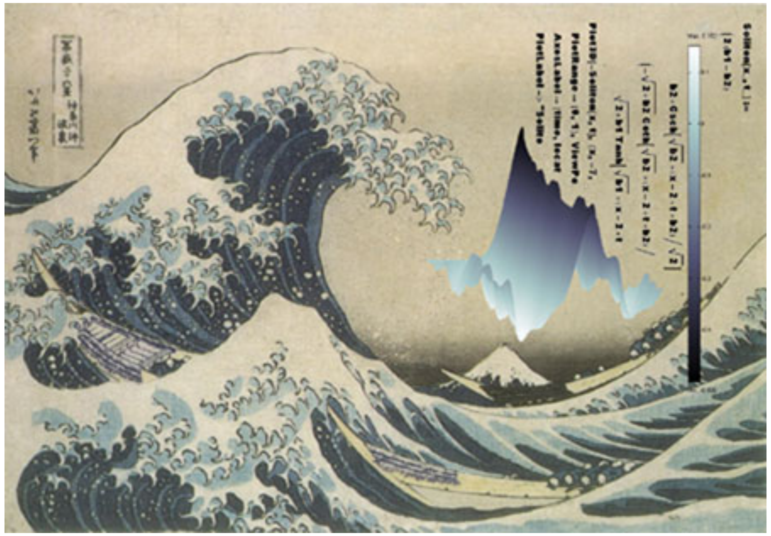
The LF scheme is FTCS – forward in time & centered in space. It preserves positivity, & thus monotonicity of shocks in the nonlinear case, but is not terribly accurate, i.e., its truncation error is only of first order in Δt and Δx.

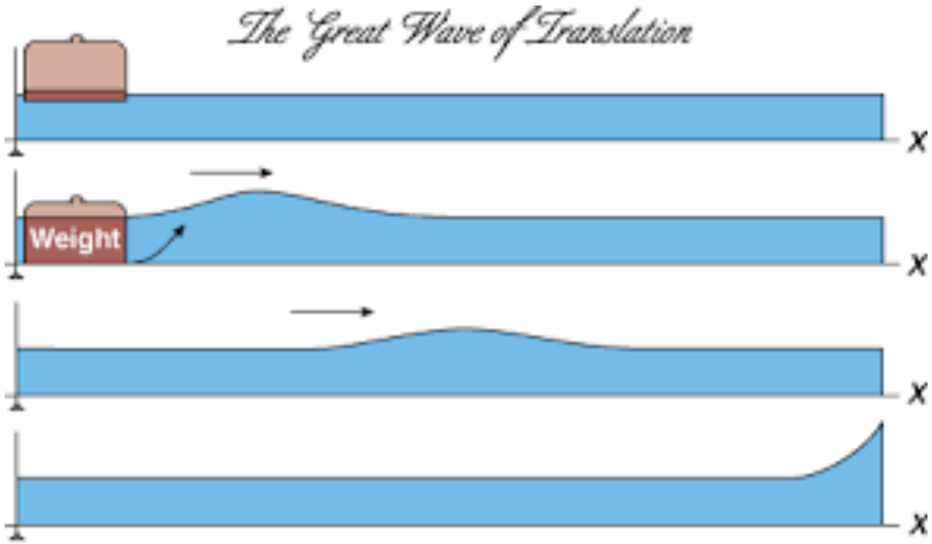
– Lax-Wendroff scheme

The LW scheme achieves second-order accuracy by also using u_i^n in the Taylor expansion of u_x, but positivity is lost; this loss leads to "ringing" or the Gibbs phenomenon in the computation of shocks.

→ The quandary of accuracy vs. monotonicity was solved in the Ph.D. thesis of Amiram Harten (RIP) with Peter, by introducing total-variation diminishing (TVD) methods. Later, Ami and Stanley Osher realized the applicability and usefulness of TVD methods in "edge detection" and hence image compression.

P. D. Lax Contributions – II, ∞-D Dynamical Systems





Not every big wave is a solitary wave nor a soliton: it must (a) be local; (b) preserve its shape and (c) "non-interact."

P. D. Lax Contributions – II, ∞-D Dynamical Systems

- **The KdV equation** – J. Scott Russell (1834) – observation + experiment
- J. Boussinesq (1877) + D.J. Korteweg & G. De Vries (1895): KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

- The KdV equation balances nonlinearity uu_x and dispersion u_{xxx} of the waves.
- N. Zabusky & M. Kruskal (1965) obtain numerical solutions and, with C.S. Gardner, J.M. Greene & R.M. Miura (1967, 1968), Kruskal finds an infinite number of constants of motion.

→ P.D. Lax (*CPAM*, 1968) associated with every solution $u = u(t)$ of $u_t = K(u)$ a linear Sturm-Liouville operator L_u and the linear PDE

$$L_t = BL - LB. \quad \text{(L)}$$

Here B_t is a one-parameter family of unitary operators, like $L_{u(t)}$, (B, L) is a **Lax pair** and (L) is the **Lax equation**.

- Many other ∞-dimensional Hamiltonian problems have these nice properties, i.e., an infinite number of invariants = {the eigenvalues of L}: nonlinear Schrödinger equation, sine-Gordon equation, Toda lattice.
- Localized coherent structures in geophysical fluid dynamics (GFD): 2-D modons and their 3-D generalizations, thermons ("hetons"), etc.

The Lax School - I

Peter David Lax

Biography MathSciNet

Ph.D. New York University 1949

Dissertation: *Nonlinear System of Hyperbolic Partial Differential Equations in Two Independent Variables*

Advisor: Kurt Otto Friedrichs

Students:
Click here to see the students listed in chronological order.

Name	School	Year	Descendants
Steven Alpern	New York University	1973	1
Satish Anjilvel	New York University	1984	
Gregory Beylkin	New York University	1982	6
Gui-Qiang Chen	Chinese Academy of Sciences	1987	21
Li-an Laurence Chen	New York University	1968	
Min Chen	New York University	1996	
Alexandre Chorin	New York University	1966	241
Melvyn Ciment	New York University	1968	
Gary Deem	New York University	1969	
Carlos De Moura	New York University	1976	8
Shlomo Engelberg	New York University	1994	
Charles Epstein	New York University	1983	6
Hector Fattorini	New York University	1965	2
Linus Foy	New York University	1962	
Porter Gerber	New York University	1968	
Michael Ghil	New York University	1975	74
Charles Goldstein	New York University	1967	
William Goodhue	New York University	1971	
Susan Hahn	New York University	1957	
Milton Halem	New York University	1968	
Amiram Harten	New York University	1974	
Brian Hayes	New York University	1994	
Reuben Hersh	New York University	1962	13
James Hyman	New York University	1976	5
Donald Isaac	New York University	1970	
Gray Jennings	New York University	1971	
Kayyunnapara Joseph	New York University	1987	2
Robert Kalaba	New York University	1958	
Spyridon Kamvissis	New York University	1991	
Barbara Keyfitz	New York University	1970	4
Arnold Lapidus	New York University	1967	
James La Vita	New York University	1967	
Charles Levermore	New York University	1982	43
George Logemann	New York University	1965	
James Moeller	New York University	1961	3
Lucien Neustadt	New York University	1960	
Sebastian Noelle	New York University	1990	16
Gideon Peyser	New York University	1957	
George Pimbley	New York University	1957	
Julian Prince	New York University	1967	
Donald Quarles, Jr.	New York University	1964	
Jeffrey Rauch	New York University	1971	35
Kennard Reed	New York University	1963	
Milton Rose	New York University	1953	
Norman Rushfield	New York University	1963	
Yiorgos Smyrlis	New York University	1989	2
Blair Swartz	New York University	1970	
Mikhail Teytel	New York University	1996	
Feiran Tian	New York University	1991	
Peter Treuenfels	New York University	1957	
Stephanos Venakides	New York University	1982	11
Homer Walker	New York University	1970	9
Burton Wendroff	New York University	1958	
Andrew Winkler	New York University	1987	
Yahan Yang	New York University	1991	

According to our current on-line database, Peter Lax has 55 students and 555 descendants.
We welcome any additional information.

The Lax School - II

Peter Lax has 55 (former) Ph.D. students and 555 descendants.

But he was labeled “the most versatile mathematician of his generation” by the Abel Prize selection committee.

And I would say that his school extends to wherever good mathematics is done and to whoever does it, …

... especially if they are as kind and charming as Peter.

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(*) Please see
Peter D. Lax: A Life in Mathematics, by M. Ghil
http://aimsciences.org/conferences/2016/Lax_biosketch-AIMS-M_Ghil_vf.pdf

Outline – Dynamical systems & climate sensitivity

- The IPCC process: results and uncertainties
- Natural climate variability as a source of uncertainties
  - sensitivity to initial data ➔ error growth
  - sensitivity to model formulation ➔ see below!
- Uncertainties and how to fix them
  - structural stability and other kinds of robustness
  - non-autonomous and random dynamical systems (NDDS & RDS)
- Two illustrative examples
  - the Lorenz convection model
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- Conclusions and references
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  - selected bibliography

Motivation

- The ***climate system*** is highly ***nonlinear and*** quite ***complex***.
- The system's ***major components*** — the atmosphere, oceans, ice sheets — ***evolve*** on many time and space scales.
- Its ***predictive understanding*** has to rely on the system's physical, chemical and biological modeling,

  but also on the thorough mathematical analysis of the models

  thus obtained: ***the forest vs. the trees***.
- The ***hierarchical modeling*** approach allows one to

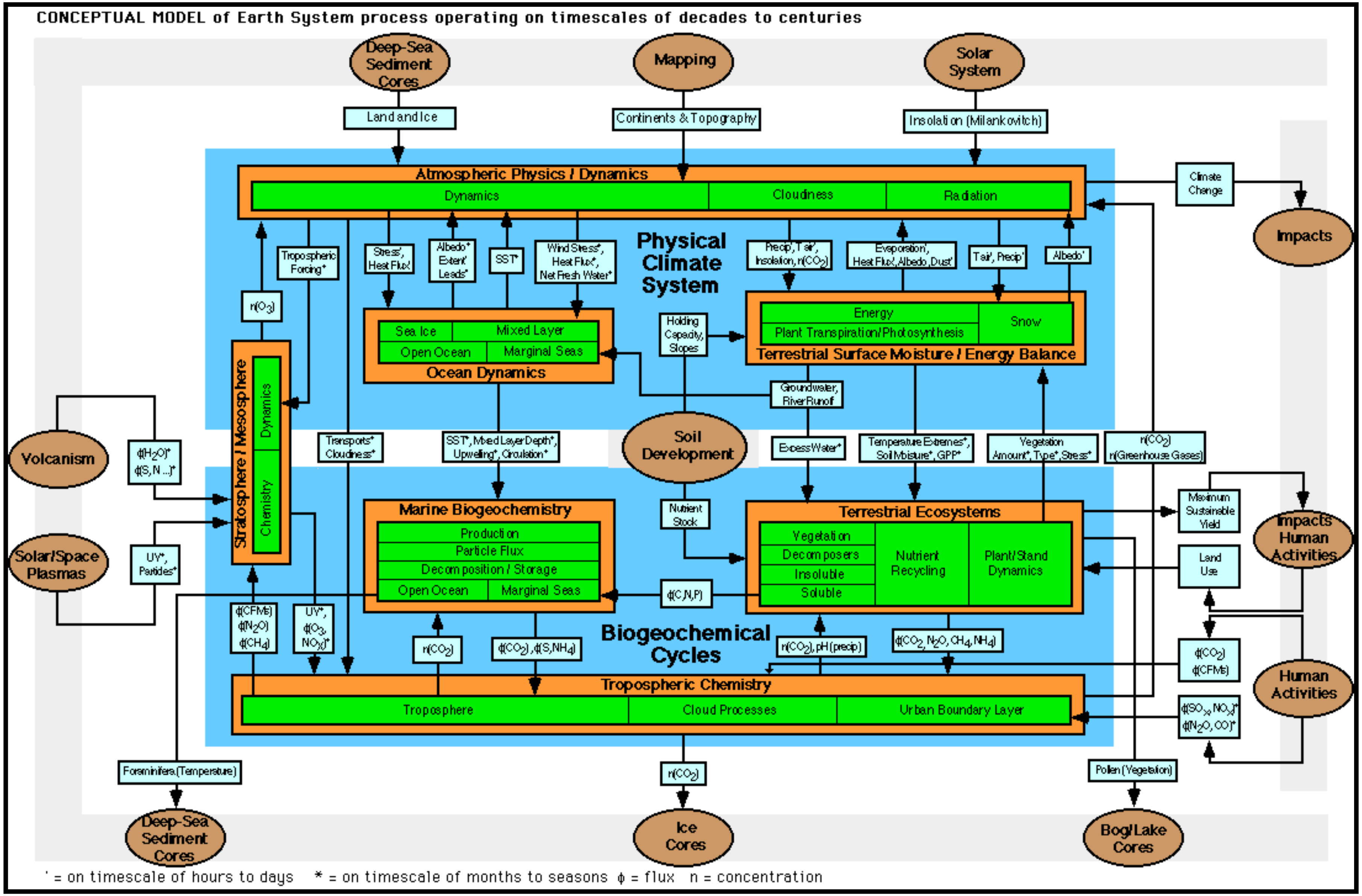
  give proper weight to the understanding provided by the

  models *vs*. their realism: back-and-forth between

  "toy" (conceptual) and ***detailed*** ("realistic") ***models***,

  and between ***models*** and ***data***.
- How do we disentangle ***natural variability*** from ***the anthropogenic forcing***: ***can we & should we, or not?***

F. Bretherton's "horrendogram" of Earth System Science



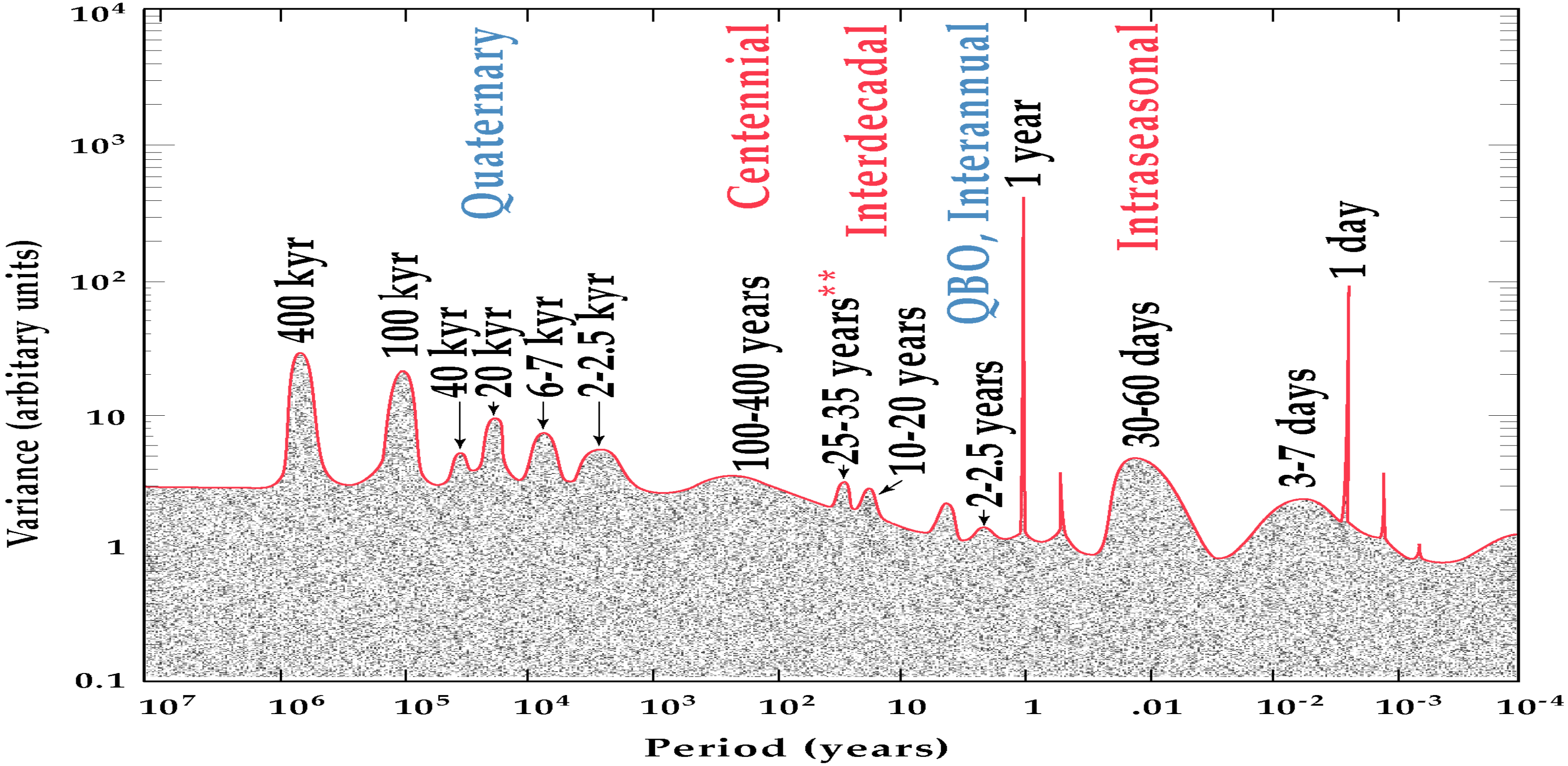
Earth System Science Overview, *NASA Advisory Council*, 1986

Composite spectrum of climate variability

Standard treatement of frequency bands:

1. High frequencies – white noise (or "colored")

2. Low frequencies – slow evolution of parameters



From **Ghil (2001, *EGEC*)**, after **Mitchell* (1976)**

*** "No known source of deterministic internal variability"**

**** 27 years – Brier (1968, *Rev. Geophys.*)**

Climate and Its Sensitivity

Let's say CO_2 doubles:

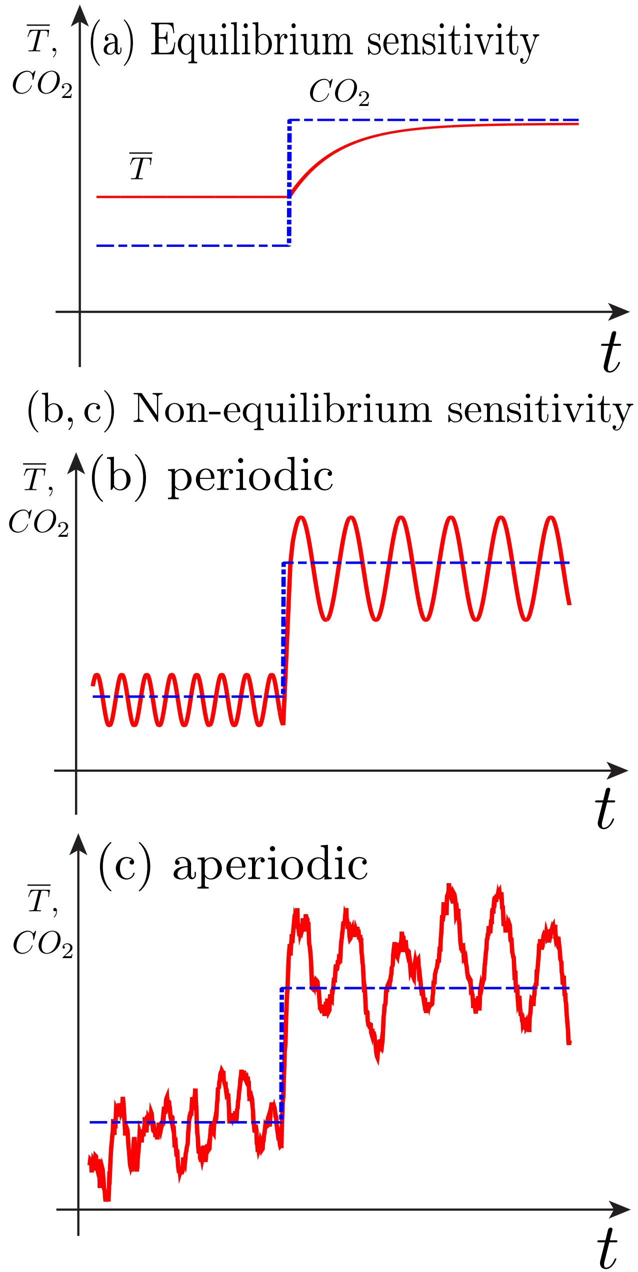
How will "climate" change?

1. Climate is in stable equilibrium (fixed point); if so, mean temperature will just shift gradually to its new equilibrium value.

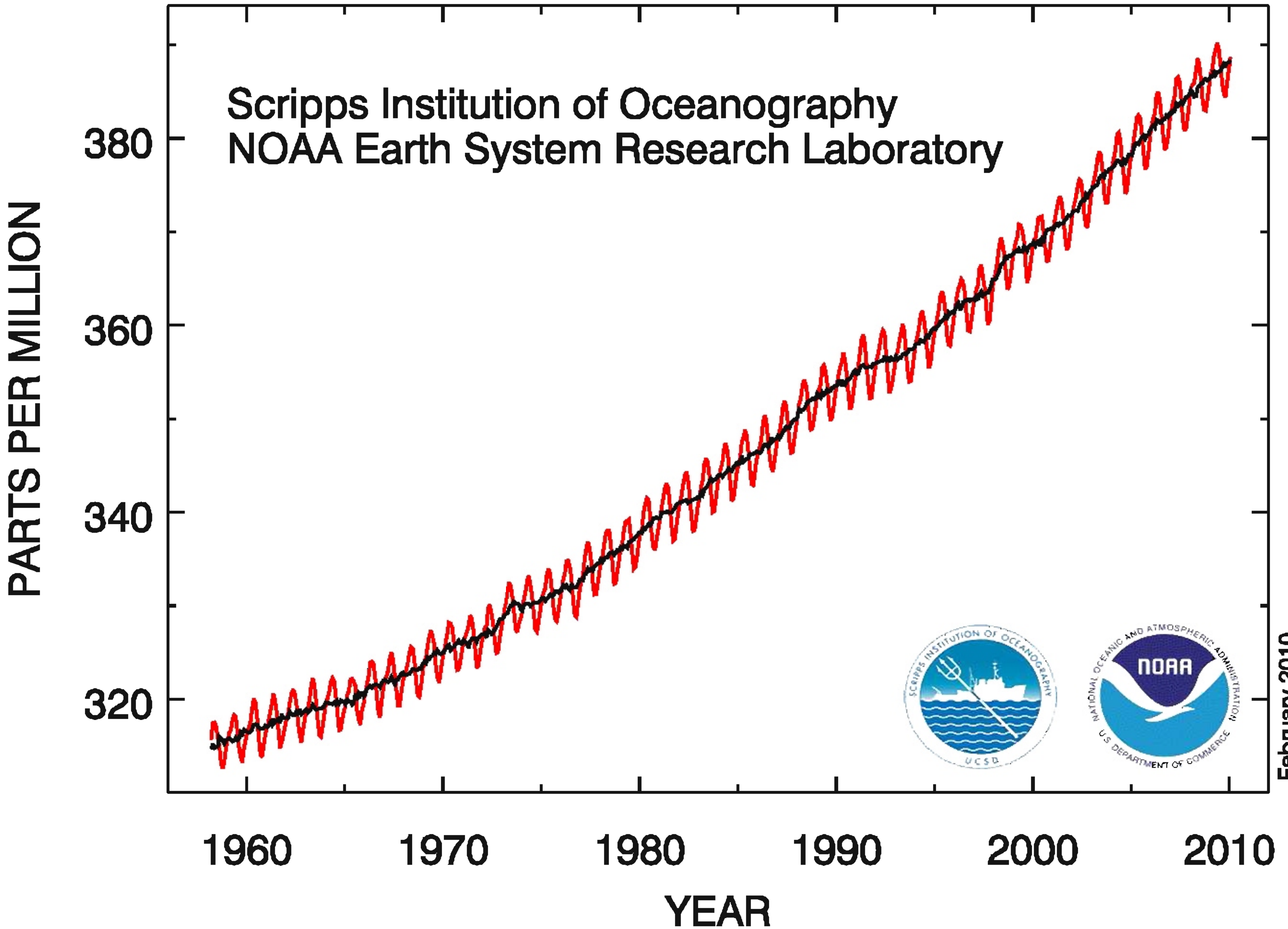
2. Climate is purely periodic; if so, mean temperature will (maybe) shift gradually to its new equilibrium value. But how will the period, amplitude and phase of the limit cycle change?

3. And how about some "real stuff" now: chaotic + random?

Ghil (in *Encycl. Global Environmental Change*, 2002)



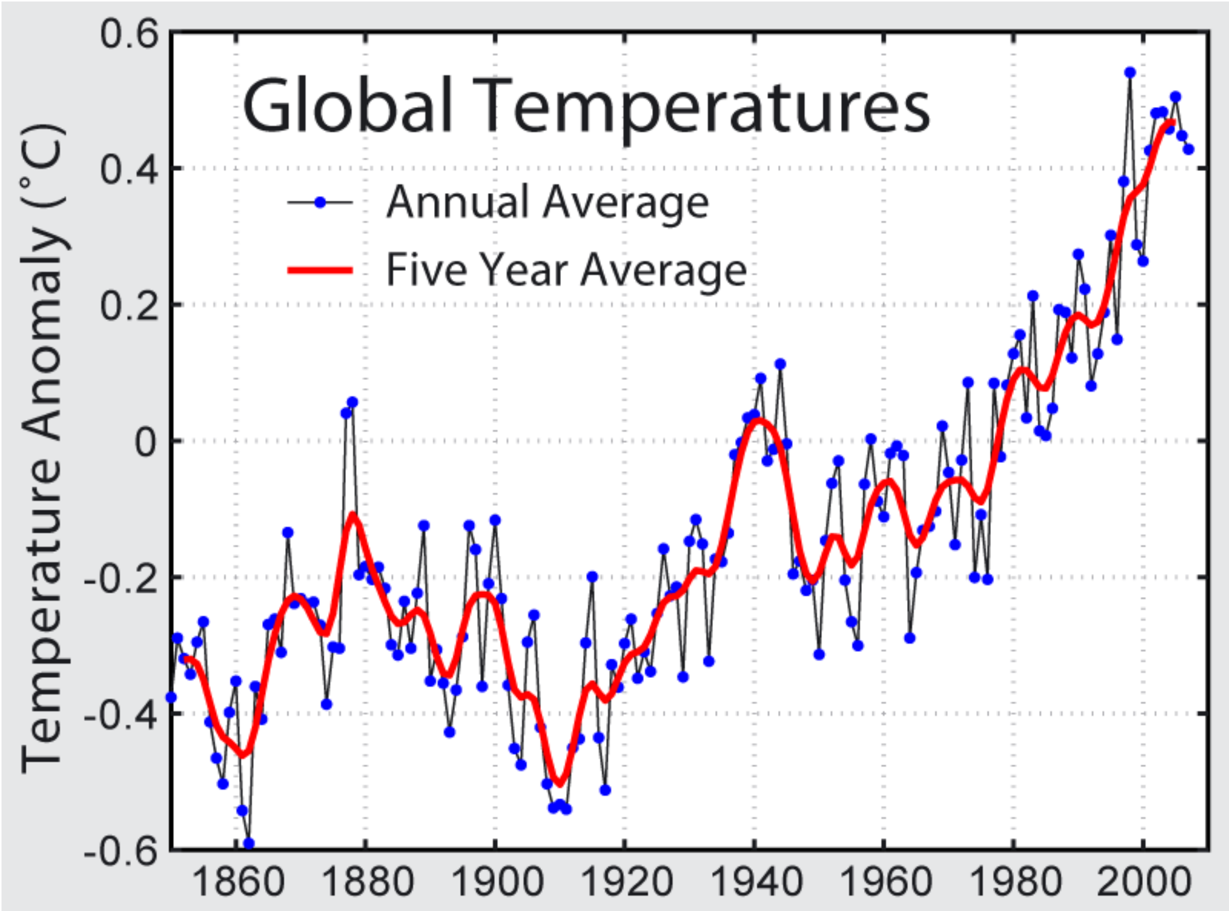
Atmospheric CO_2 at Mauna Loa Observatory



Temperatures and GHGs

Greenhouse gases (GHGs) go up,
temperatures go up:

It's gotta do with us, at least a bit,
doesn't it?



Wikicommons, from
Hansen *et al*. (*PNAS*, 2006);
see also http://data.giss.nasa.gov/
gistemp/graphs/

Global warming and its socio-economic impacts

Temperatures rise:

- What about impacts?
- How to adapt?

The answer, my friend, is blowing in the wind, *i.e.*, it depends on the accuracy and reliability of the forecast …

Source : IPCC (2007), *AR4, WGI, SPM*

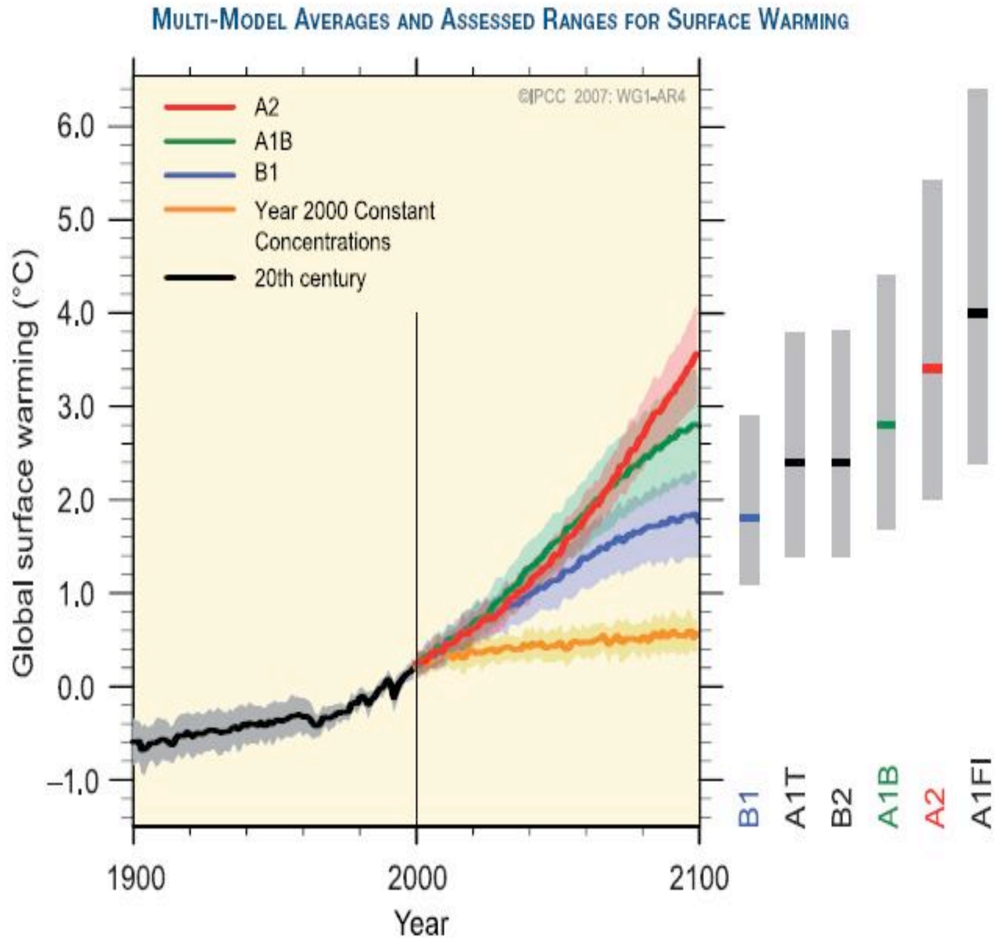


Figure SPM.5. *Solid lines are multi-model global averages of surface warming (relative to 1980–1999) for the scenarios A2, A1B and B1, shown as continuations of the 20th century simulations. Shading denotes the ±1 standard deviation range of individual model annual averages. The orange line is for the experiment where concentrations were held constant at year 2000 values. The grey bars at right indicate the best estimate (solid line within each bar) and the* **likely** *range assessed for the six SRES marker scenarios. The assessment of the best estimate and* **likely** *ranges in the grey bars includes the AOGCMs in the left part of the figure, as well as results from a hierarchy of independent models and observational constraints. {Figures 10.4 and 10.29}*

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  - sensitivity to initial data ➔ error growth
  - sensitivity to model formulation ➔ see below!
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So what's it gonna be like, by 2100?

Table SPM.2. *Recent trends, assessment of human influence on the trend and projections for extreme weather events for which there is an observed late-20th century trend. {Tables 3.7, 3.8, 9.4; Sections 3.8, 5.5, 9.7, 11.2–11.9}*

Phenomenon[a] and direction of trend	Likelihood that trend occurred in late 20th century (typically post 1960)	Likelihood of a human contribution to observed trend[b]	Likelihood of future trends based on projections for 21st century using SRES scenarios
Warmer and fewer cold days and nights over most land areas	*Very likely*[c]	*Likely*[d]	*Virtually certain*[d]
Warmer and more frequent hot days and nights over most land areas	*Very likely*[e]	*Likely (nights)*[d]	*Virtually certain*[d]
Warm spells/heat waves. Frequency increases over most land areas	*Likely*	*More likely than not*[f]	*Very likely*
Heavy precipitation events. Frequency (or proportion of total rainfall from heavy falls) increases over most areas	*Likely*	*More likely than not*[f]	*Very likely*
Area affected by droughts increases	*Likely* in many regions since 1970s	*More likely than not*	*Likely*
Intense tropical cyclone activity increases	*Likely* in some regions since 1970	*More likely than not*[f]	*Likely*
Increased incidence of extreme high sea level (excludes tsunamis)[g]	*Likely*	*More likely than not*[f,h]	*Likely*[i]

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Can we, nonlinear dynamicists, help?

The uncertainties
might be ***intrinsic***,
rather than mere
"tuning problems"

If so, maybe
stochastic structural stability could help!

Might fit in nicely with
recent taste for
"stochastic
parameterizations"

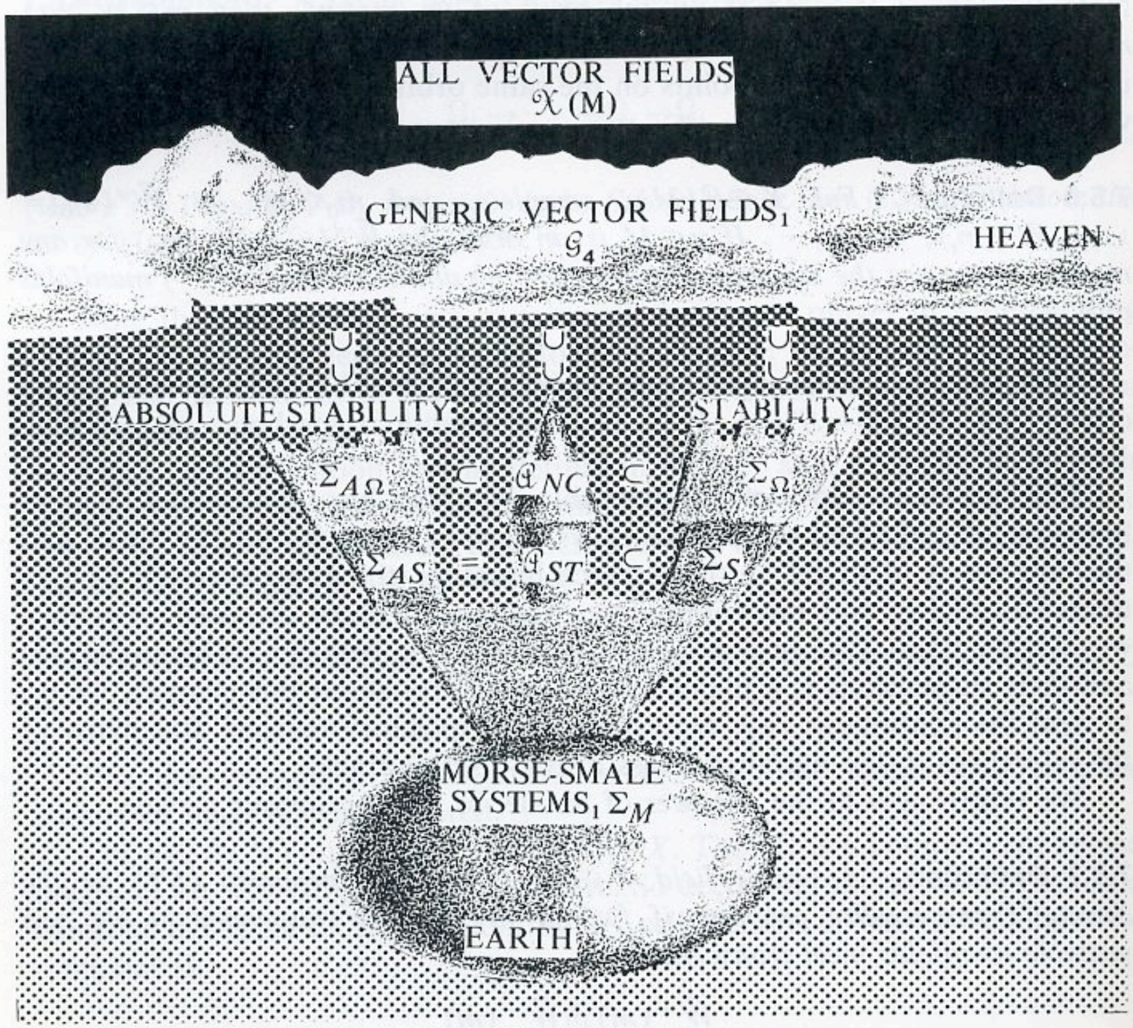


Figure 7.5-1. The three towers of differentiable dynamics.

The DDS dream of structural stability (from *Abraham & Marsden*, 1978)

Non-autonomous Dynamical Systems

A linear, dissipative, forced example: *forward vs. pullback attraction*

Consider the scalar, linear ordinary differential equation (ODE)

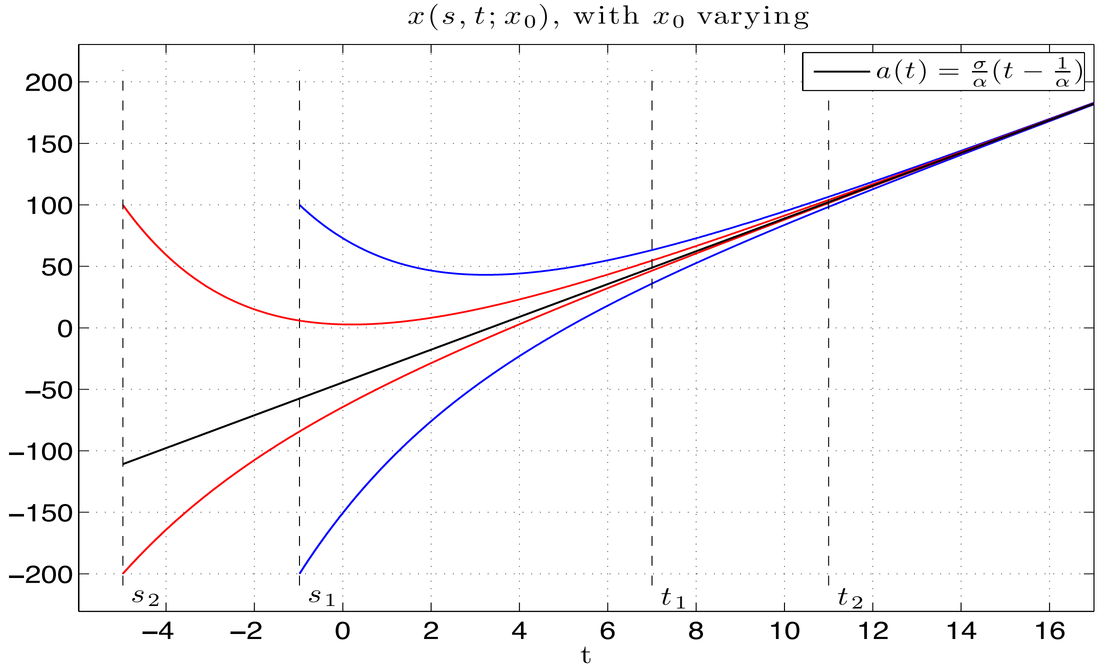
$$\dot{x} = -\alpha x + \sigma t\,, \quad \alpha > 0\,,\ \sigma > 0\,.$$

The autonomous part of this ODE, $\dot{x} = -\alpha x$, is dissipative and all solutions $x(t;x_0) = x(t;x(0) = x_0)$ converge to 0 as $t \to +\infty$.

What about the non-autonomous, forced ODE? As the energy being put into the system by the forcing is dissipated, we expect things to change in time. In fact, if we "pull back" far enough, replace $x(t;x_0)$ by $x(s,t;x_0) = x(s,t;x(s) = x_0)$,

and let $s \to -\infty$, we get th
pullback attractor $a = a(t)$
in the figure,

$$a(t) = \frac{\sigma}{\alpha}\left(t - \frac{1}{\alpha}\right).$$



Random Dynamical Systems (RDS), I - RDS theory

This theory is the counterpart for randomly forced dynamical systems (RDS) of the *geometric theory* of ordinary differential equations (ODEs). It allows one to treat stochastic differential equations (SDEs) — and more general systems driven by noise — as flows in (phase space)$\times$(probability space).
SDE$\sim$ODE, RDS$\sim$DDS, L. Arnold (1998)$\sim$V.I. Arnol'd (1983).

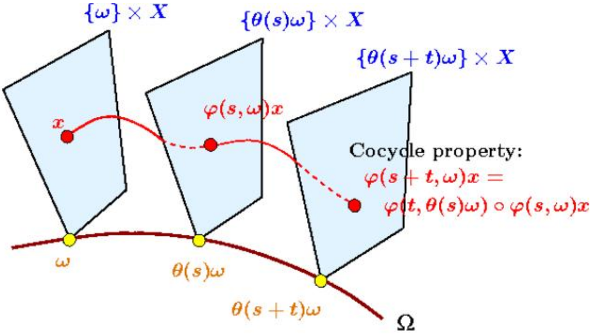
Setting:

(i) A phase space X. **Example**: $\mathbb{R}^n$.

(ii) A probability space $(\Omega, \mathcal{F}, \mathbb{P})$. **Example**: The Wiener space $\Omega = C_0(\mathbb{R}; \mathbb{R}^n)$ with Wiener measure $\mathbb{P}$.

(iii) A model of the noise $\theta(t) : \Omega \to \Omega$ that preserves the measure $\mathbb{P}$, i.e. $\theta(t)\mathbb{P} = \mathbb{P}$; θ is called the driving system.
Example: $W(t, \theta(s)\omega) = W(t + s, \omega) - W(s, \omega)$;
it starts the noise at s instead of $t = 0$.

(iv) A mapping $\varphi : \mathbb{R} \times \Omega \times X \to X$ with the cocycle property.
Example: The solution operator of an SDE.



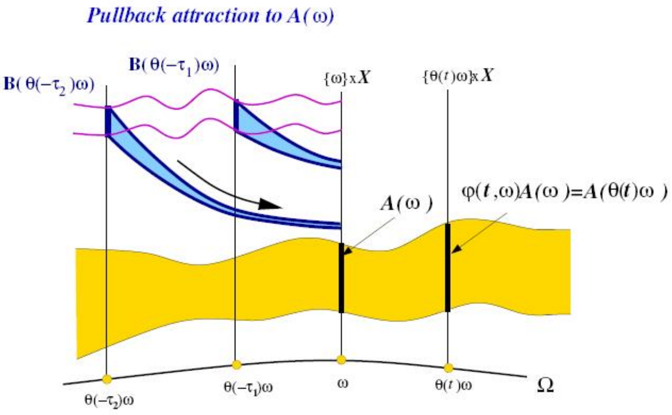
- φ **is a random dynamical system (RDS)**
- $\Theta(t)(x,\omega) = (\theta(t)\omega, \varphi(t,\omega)x)$ **is a flow on the bundle**

RDS, III- Random attractors (RAs)

A random attractor $\mathcal{A}(\omega)$ is both *invariant* and "pullback" *attracting*:

(a) **Invariant**: $\varphi(t,\omega)\mathcal{A}(\omega) = \mathcal{A}(\theta(t)\omega)$.

(b) **Attracting**: $\forall B \subset X$, $\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\theta(-t)\omega)B, \mathcal{A}(\omega)) = 0$ a.s.



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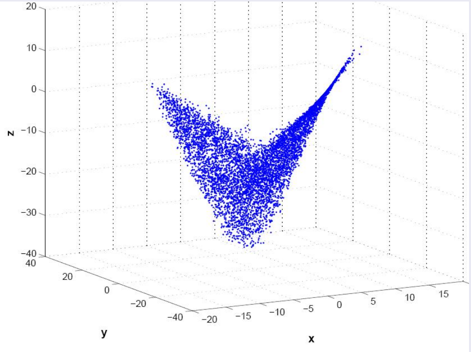
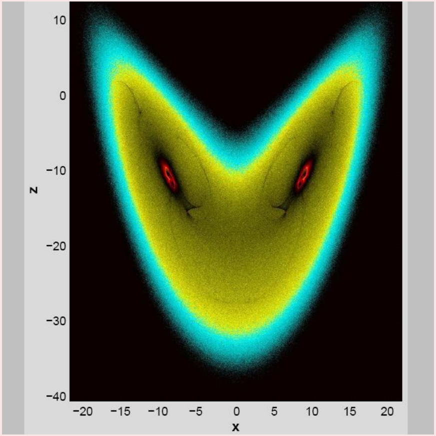
Random attractor of the stochastic Lorenz system

Snapshot of the random attractor (RA)



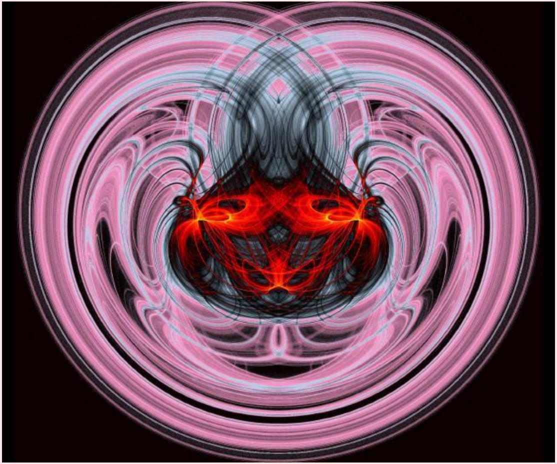
- A snapshot of the RA, $\mathcal{A}(\omega)$, computed at a fixed time t and for the same realization ω; it is made up of points transported by the stochastic flow, from the remote past $t - T$, $T >> 1$.
- We use multiplicative noise in the deterministic Lorenz model, with the classical parameter values $b = 8/3$, $\sigma = 10$, and $r = 28$.
- Even computed pathwise, this object supports meaningful statistics.

Sample measures supported by the R.A.



- We compute the probability measure on the R.A. at some fixed time t, and for a fixed realization ω. We show a "projection", $\int \mu_\omega(x, y, z)\mathrm{d}y$, with multiplicative noise: $\mathrm{d}x_i$=Lorenz$(x_1, x_2, x_3)\mathrm{d}t + \alpha\ x_i \mathrm{d}W_t; i \in \{1, 2, 3\}$.
- 10 million of initial points have been used for this picture!

Sample measure supported by the R.A.



- Still 1 Billion I.D., and $\alpha = 0.5$. Another one?

Sample measure supported by the R.A.

Sample measures evolve with time.

- Recall that these sample measures are the frozen statistics at a time *t* for a realization ω.
- How do these frozen statistics evolve with time?
- Action!

A day in the life of the Lorenz (1963) model's random attractor, or LORA for short;
see SI in Chekroun, Simonnet & Ghil (2011, *Physica D*)

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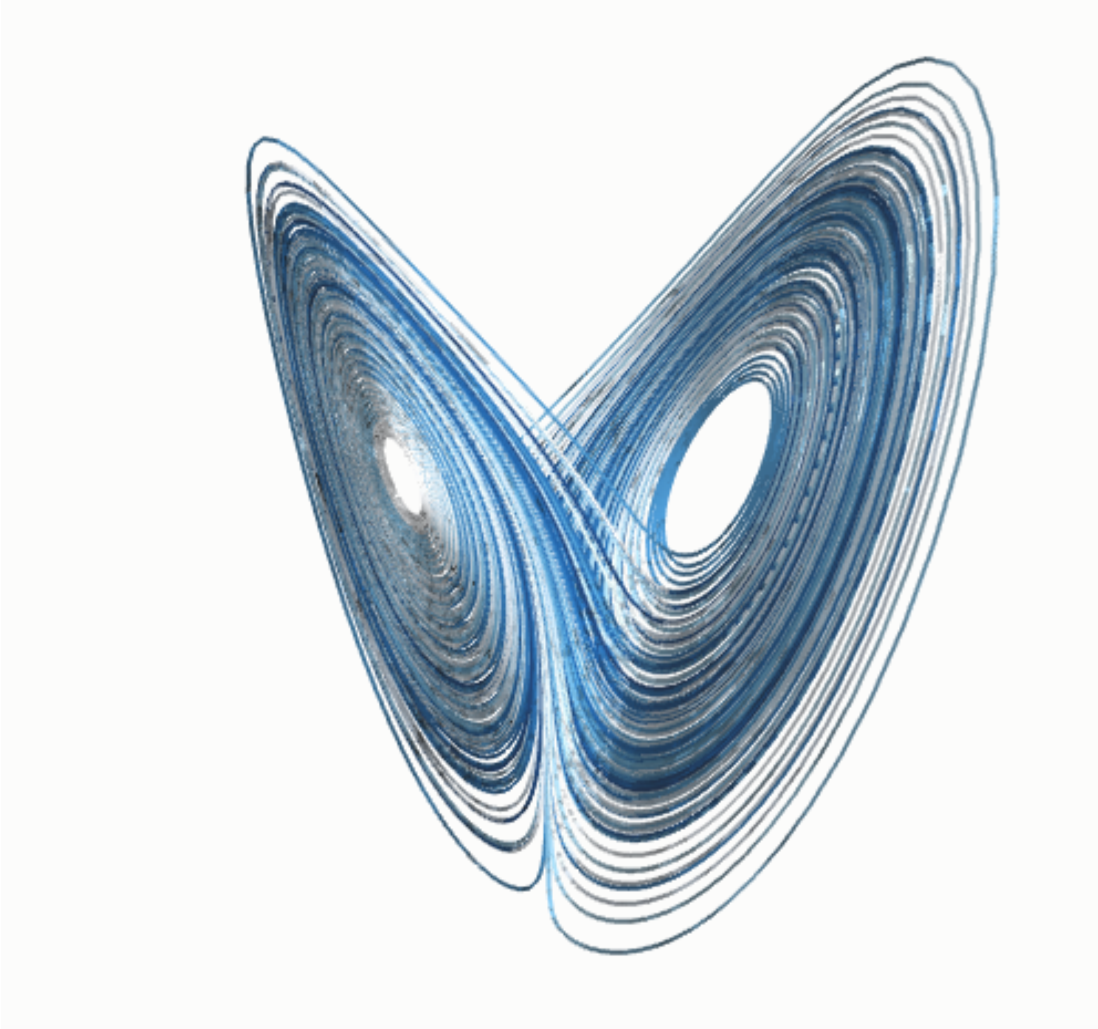
Classical Strange Attractor

Physically closed system, modeled mathematically as autonomous system: neither deterministic (anthropogenic) nor random (natural) forcing.

The attractor is strange, but still fixed in time ~ “irrational” number.

Climate sensitivity ~ change in the average value (first moment) of the coordinates *(x, y, z)* as a parameter λ changes.



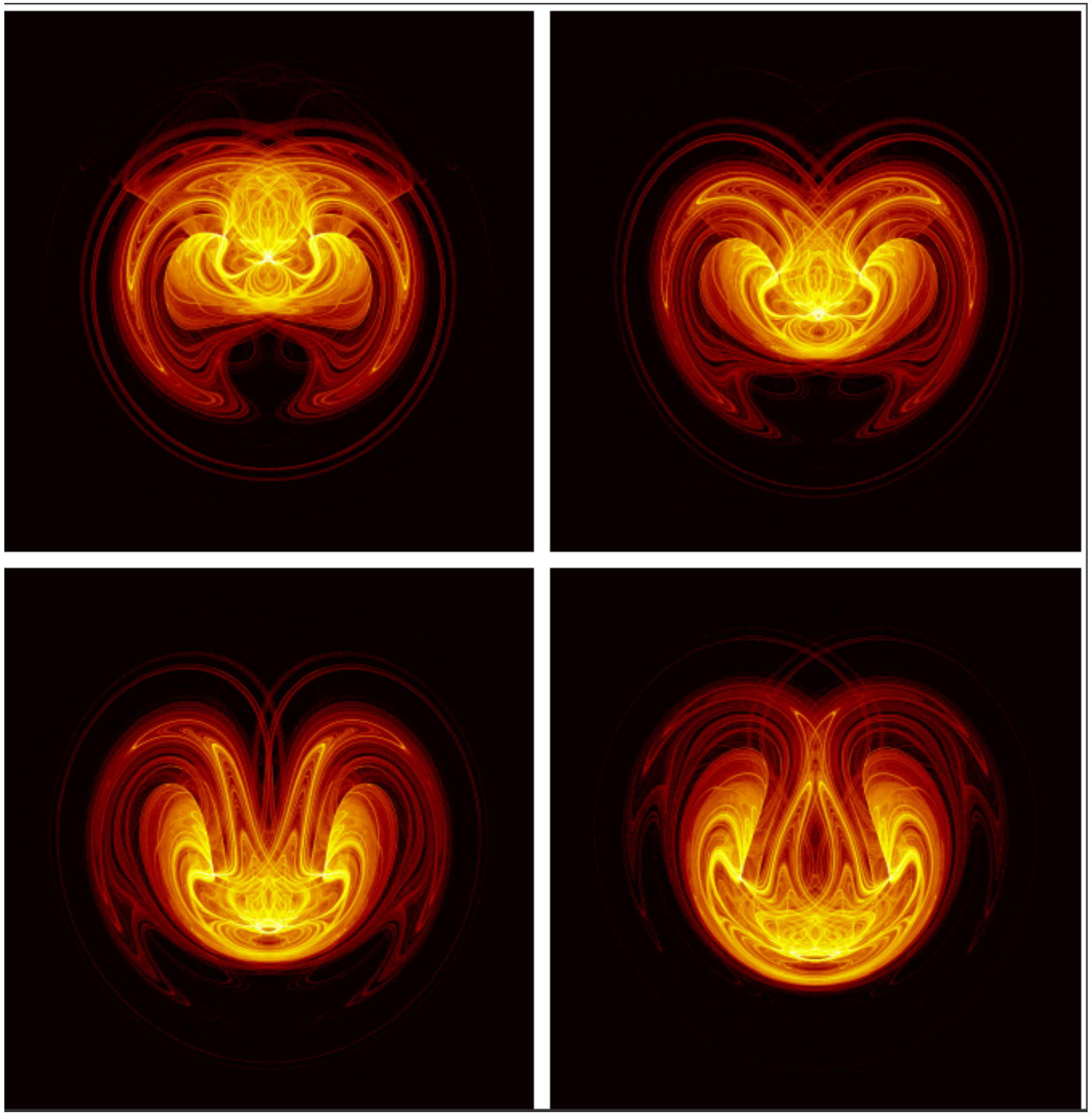
Random Attractor

Physically open system, modeled mathematically as non-autonomous system: allows for deterministic (anthropogenic) as well as random (natural) forcing.

The attractor is "pullback" and evolves in time ~ "imaginary" or "complex" number.

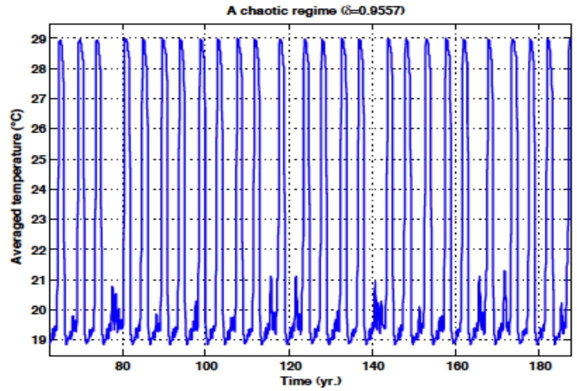
Climate sensitivity ~ change in the statistical properties (first and higher-order moments) of the attractor as one or more parameters (λ, μ, …) change.

Ghil (*Encyclopedia of Atmospheric Sciences*, 2nd ed., 2012)



Parameter dependence – I

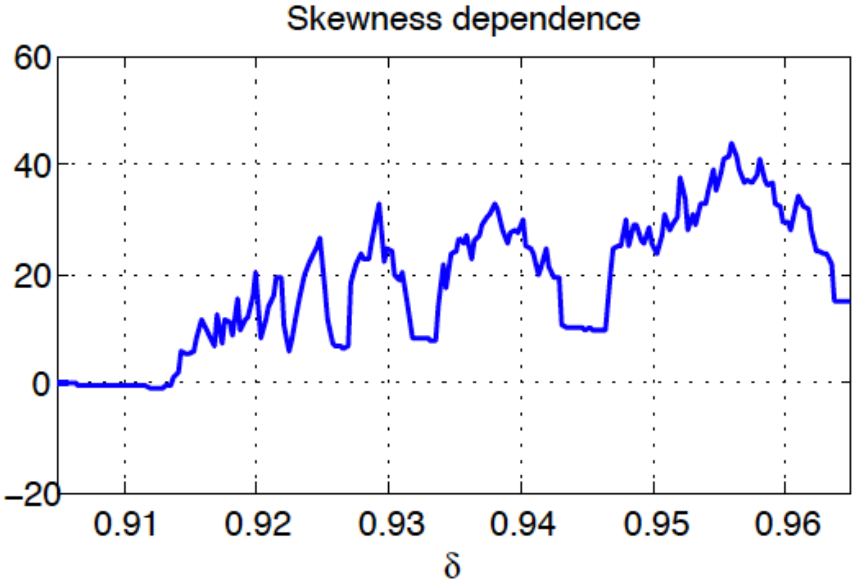
$\delta = 0.9557$

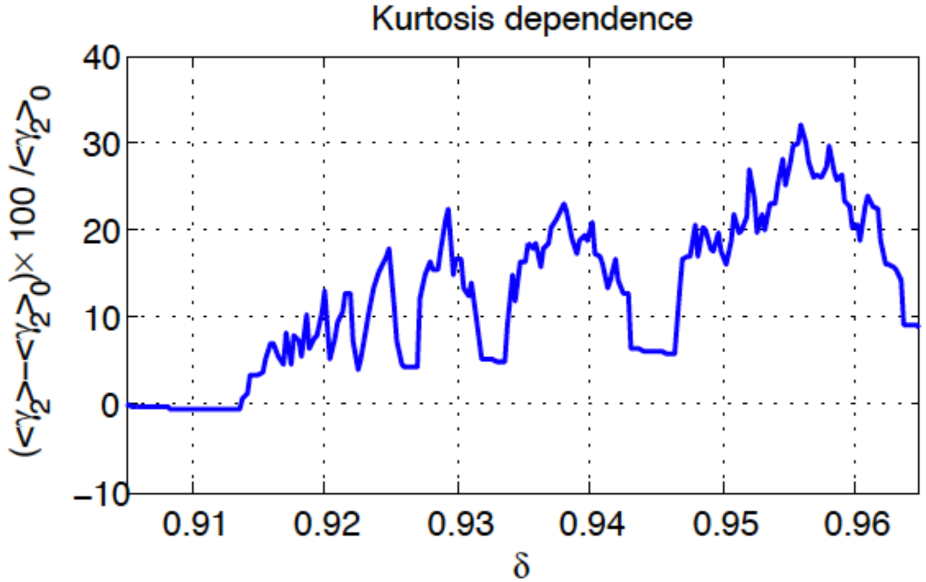


It can be smooth or it can be rough:
Niño-3 SSTs from intermediate coupled model for ENSO (Jin, Neelin & Ghil, 1994, 1996)

Skewness & kurtosis of the SSTs:
time series of 4000 years,

$$\Delta\delta = 3 \cdot 10^{-4}$$





M. Chekroun (work in progress)

Sample measures for an NDDE model of ENSO

The Galanti-Tziperman (GT) model (*JAS*, 1999)

$$\frac{dT}{dt} = -\epsilon_T T(t) - M_0(T(t) - T_{sub}(h(t))),$$

$$\begin{aligned} h(t) = M_1 e^{-\epsilon_m(\tau_1+\tau_2)} h(t-\tau_1-\tau_2) \\ -M_2\tau_1 e^{-\epsilon_m(\frac{\tau_1}{2}+\tau_2)}\mu(t-\tau_2-\frac{\tau_1}{2})T(t-\tau_2-\frac{\tau_1}{2}) \\ +M_3\tau_2 e^{-\epsilon_m\frac{\tau_2}{2}}\mu(t-\frac{\tau_2}{2})T(t-\frac{\tau_2}{2}). \end{aligned}$$

Neutral delay-differential equation (NDDE), derived from Cane-Zebiak and Jin-Neelin models for ENSO: *T* is East-basin SST and *h* is thermocline depth.

Seasonal forcing given by $\mu(t) = 1 + \epsilon\cos(\omega t + \phi)$. The pullback attractor and its invariant measures were computed.

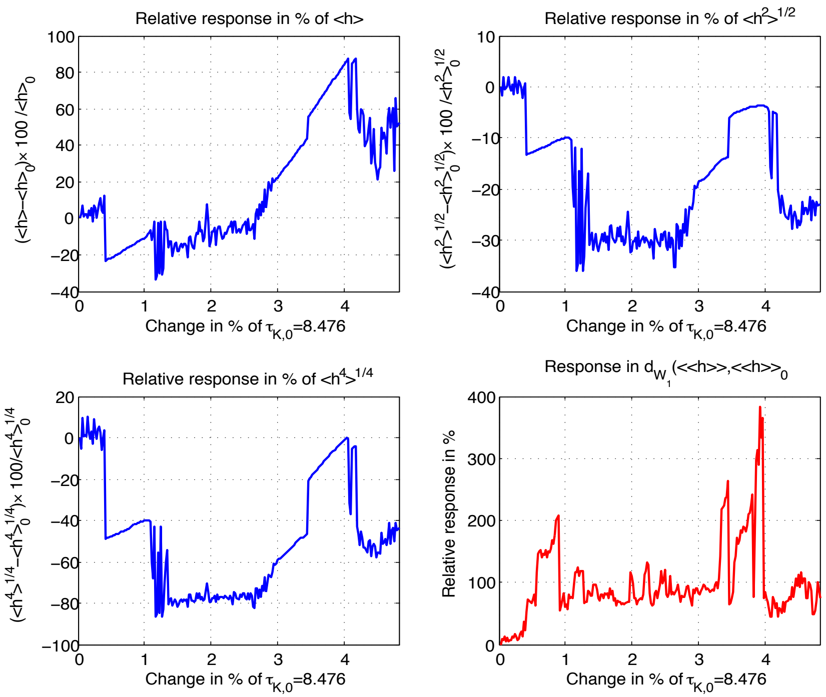


Figure shows the changes in the mean, 2nd & 4th moment of *h*(*t*), along with the Wasserstein distance d_W, for changes of 0–5% in the delay parameter $\tau_{\kappa,0}$

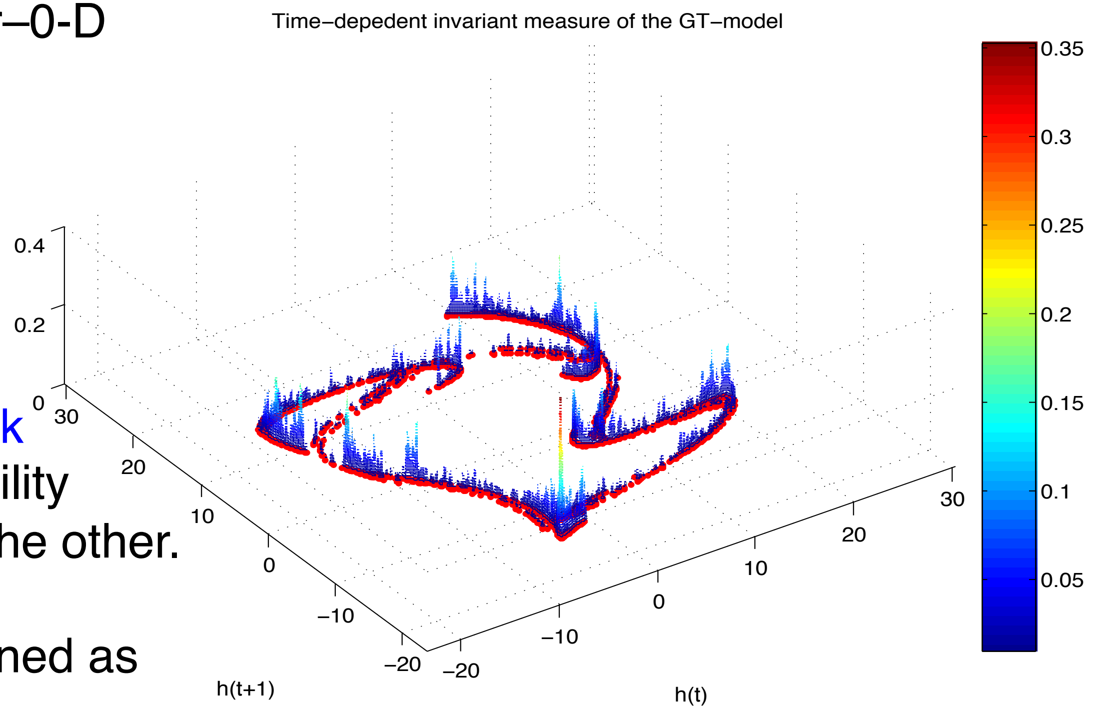
Note intervals of both smooth & rough dependence!

Pullback attractor and invariant measure of the GT model

The time-dependent pullback attractor of the GT model supports an invariant measure $\nu = \nu(t)$, whose density is plotted in 3-D perspective.

The plot is in delay coordinates *h*(*t*+1) *vs.* *h*(*t*) and the density is highly concentrated along 1-D filaments and, furthermore, exhibits sharp, near–0-D peaks on these filaments.

The Wasserstein distance d_W between one such configuration, at given parameter values, and another one, at a different set of values, is proportional to the work needed to move the total probability mass from one configuration to the other.

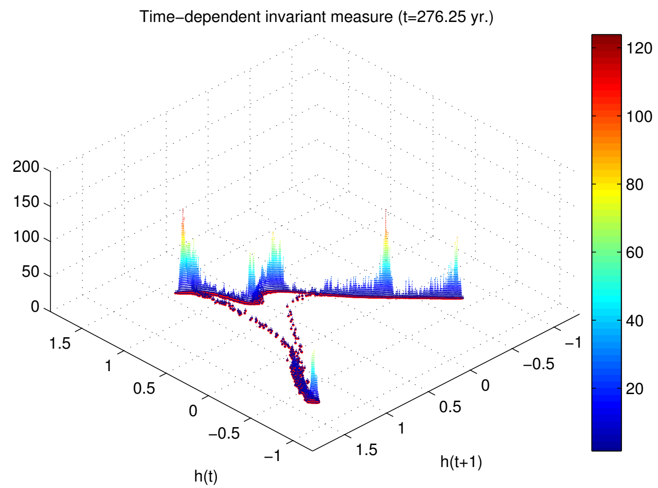


Climate sensitivity γ can be defined as

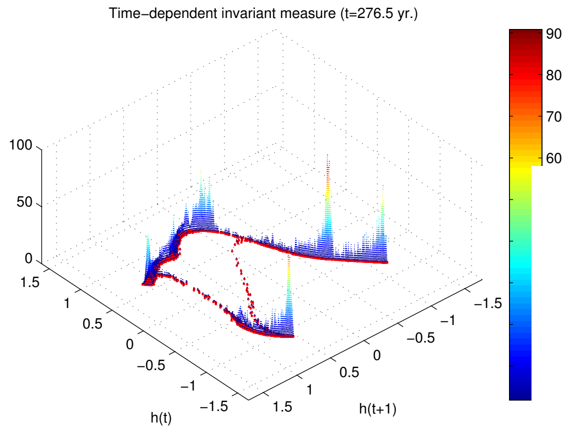
$$\gamma = \partial d_{\mathrm{W}} / \partial \tau$$

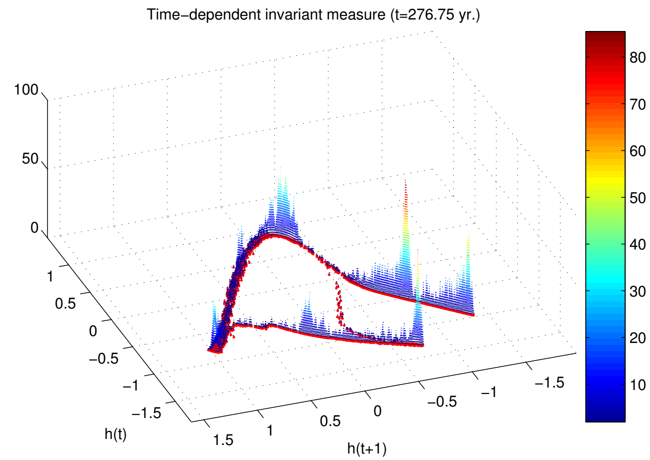
How to define climate sensitivity or, What happens when there's natural variability?

This definition allows us to watch how "the earth moves," as it is affected by both natural and anthropogenic forcing, in the presence of natural variability, which includes both chaotic & random behavior:



$$\gamma = \partial d_{\mathrm{W}} / \partial \tau$$





Outline

- The IPCC process: results and uncertainties
- Natural climate variability as a source of uncertainties
  - sensitivity to initial data ➔ error growth
  - sensitivity to model formulation ➔ see below!
- Uncertainties and how to fix them
  - structural stability and other kinds of robustness
  - non-autonomous and random dynamical systems (NDDS & RDS)
- Two illustrative examples
  - the Lorenz convection model
  - an El Niño–Southern Oscillation (ENSO) model
- Linear response theory and climate sensitivity
- Conclusions and references
  - natural variability and anthropogenic forcing: the "grand unification"
  - selected bibliography

Concluding remarks, I – RDS and RAs

Summary

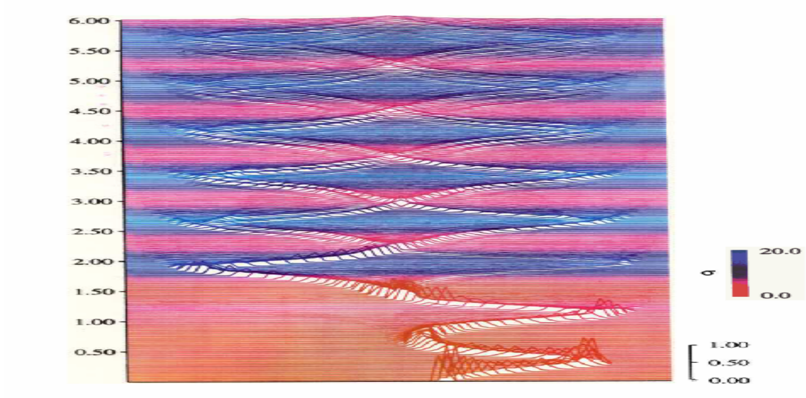
- A change of paradigm from **closed, autonomous systems** to **open, non-autonomous ones.**
- Random attractors are (i) spectacular, (ii) useful, and (iii) just starting to be explored for climate applications.

Work in progress

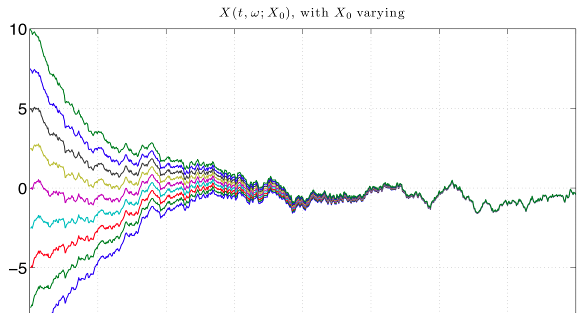
- Study the effect of specific stochastic parametrizations on model robustness.
- Applications to intermediate models and GCMs.
- Implications for climate sensitivity.
- Implications for predictability?

Yet another (grand?) unification

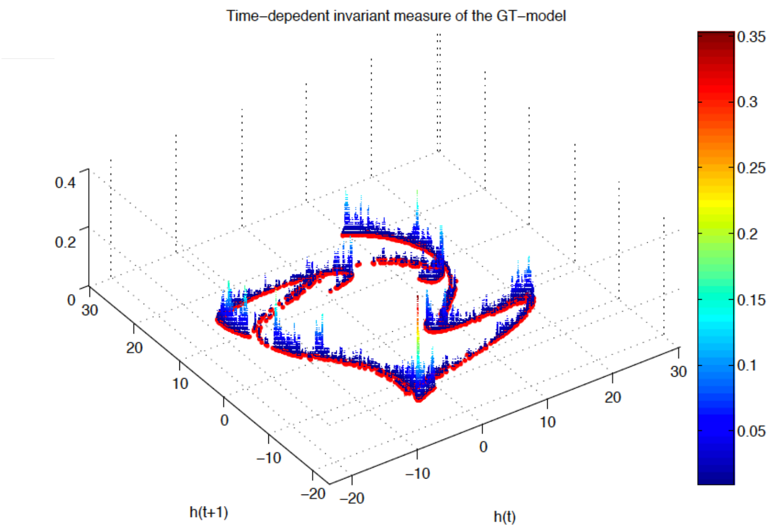
Lorenz (*JAS*, 1963)
Climate is deterministic and autonomous,
    but highly nonlinear.
Trajectories diverge exponentially,
    forward asymptotic PDF is multimodal.



Hasselmann (*Tellus*, 1976)
Climate is stochastic and noise-driven,
    but quite linear.
Trajectories decay back to the mean,
    forward asymptotic PDF is unimodal.



Grand unification (?)
Climate is deterministic + stochastic,
    as well as highly nonlinear.
Internal variability and forcing interact
    strongly, change and sensitivity
    refer to both mean and higher moments.



Concluding remarks, II – Climate change & climate sensitivity

What do we know?

- It's getting warmer.
- We do contribute to it.
- So we should act as best we know and can!

What do we know less well?

- By how much?
  - Is it getting warmer …
  - Do we contribute to it …
- How does the climate system (atmosphere, ocean, ice, etc.) really work?
- How does natural variability interact with anthropogenic forcing?

What to do?

- Better understand the system and its forcings.
- Explore the models', and the system's, robustness and sensitivity
  - stochastic structural and statistical stability!
  - linear response = response function + susceptibility function!!

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Climate change & climate sensitivity

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Reserve slides

Unfortunately, things aren’t all that easy!

What to do?

Try to achieve better interpretation of, and agreement between, models …

Ghil, M., 2002: Natural climate variability, in *Encyclopedia of Global Environmental Change*, T. Munn (Ed.), Vol. 1, Wiley

Natural variability introduces additional complexity into the anthropogenic climate change problem

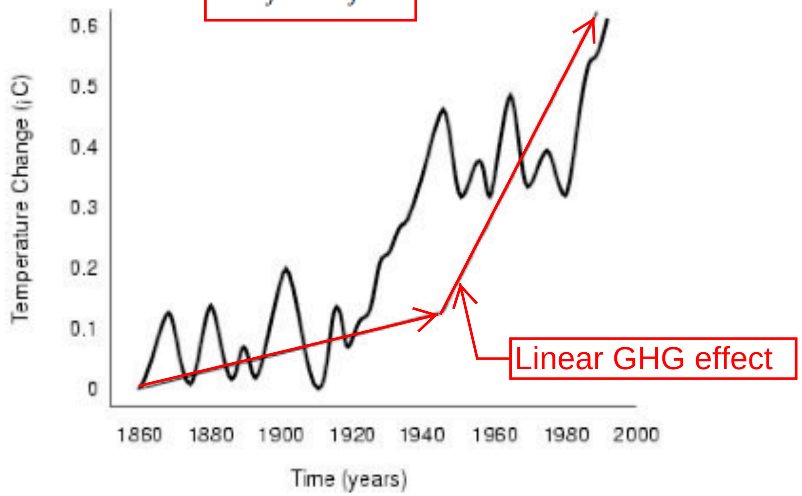
The most common interpretation of observations and GCM simulations of climate change is still in terms of a scalar, linear Ordinary Differential Equation (ODE)

$$c\frac{dT}{dt} = -kT + Q$$

$k = \sum k_i$ – feedbacks (+ve and -ve)

$Q = \sum Q_j$ – sources & sinks

$Q_j = Q_j(t)$



Linear response to CO_2 vs. observed change in T

Hence, we need to consider instead a system of nonlinear Partial Differential Equations (PDEs), with parameters and multiplicative, as well as additive forcing (deterministic + stochastic)

$$\frac{d\mathbf{X}}{dt} = \mathbf{N}(\mathbf{X}, t, \mu, \beta)$$

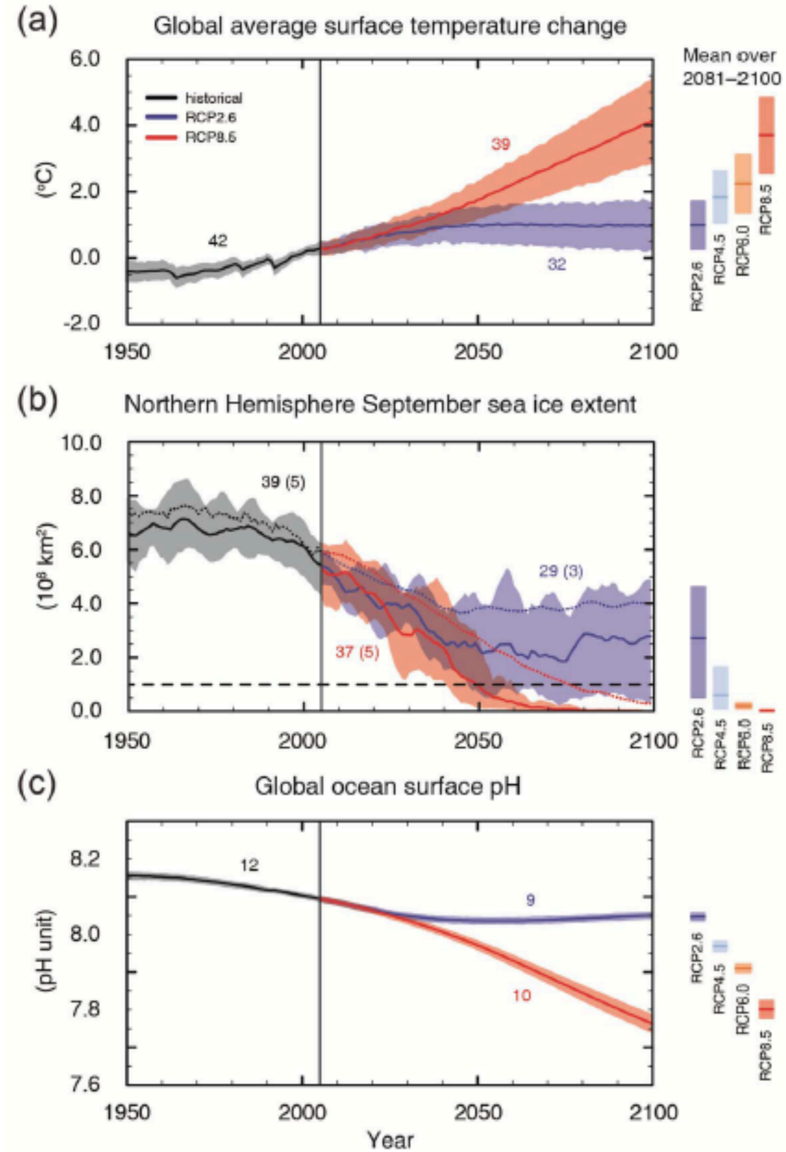
Global warming and its socio-economic impacts– II

Temperatures rise:

- What about impacts?
- How to adapt?

AR5 vs. AR4

A certain air of *déjà vu*:
GHG “scenarios” have been replaced by “representative concentration pathways” (RCPs), more dire predictions, but the uncertainties remain.



Source : IPCC (2013), AR5, WGI, SPM

letters to nature

Nature **350**, 324 - 327 (1991); doi:10.1038/350324a0

Interdecadal oscillations and the warming trend in global temperature time series

M. Ghil & R. Vautard

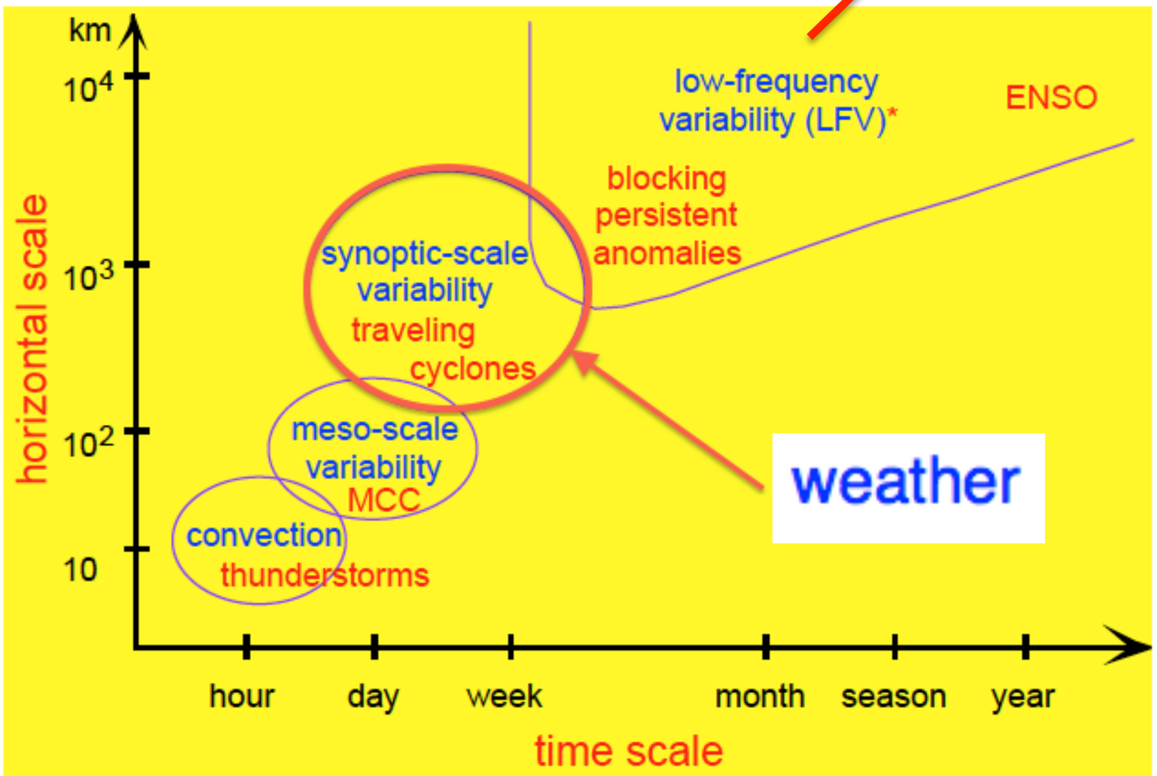
THE ability to distinguish a warming trend from natural variability is critical for an understanding of the climatic response to increasing greenhouse-gas concentrations. Here we use singular spectrum analysis[1] to analyse the time series of global surface air tem-peratures for the past 135 years[2], allowing a secular warming trend and a small number of oscillatory modes to be separated from the noise. The trend is flat until 1910, with an increase of 0.4 °C since then. The oscillations exhibit interdecadal periods of 21 and 16 years, and interannual periods of 6 and 5 years. The interannual oscillations are probably related to global aspects of the El Niño-Southern Oscillation (ENSO) phenomenon[3]. The interdecadal oscillations could be associated with changes in the extratropical ocean circulation[4]. The oscillatory components have combined (peak-to-peak) amplitudes of 0.2 °C, and therefore limit our ability to predict whether the inferred secular warming trend of 0.005 $°C yr^{-1}$ will continue. This could postpone incontrovertible detection of the greenhouse warming signal for one or two decades.

Multiple scales of motion: Space-time organization

Observations

climate

- The most active scales lie along a diagonal in this space *vs.* time plot.
- Why this is so is far from clear as of now.
- We'll deal with weather first, then climate.

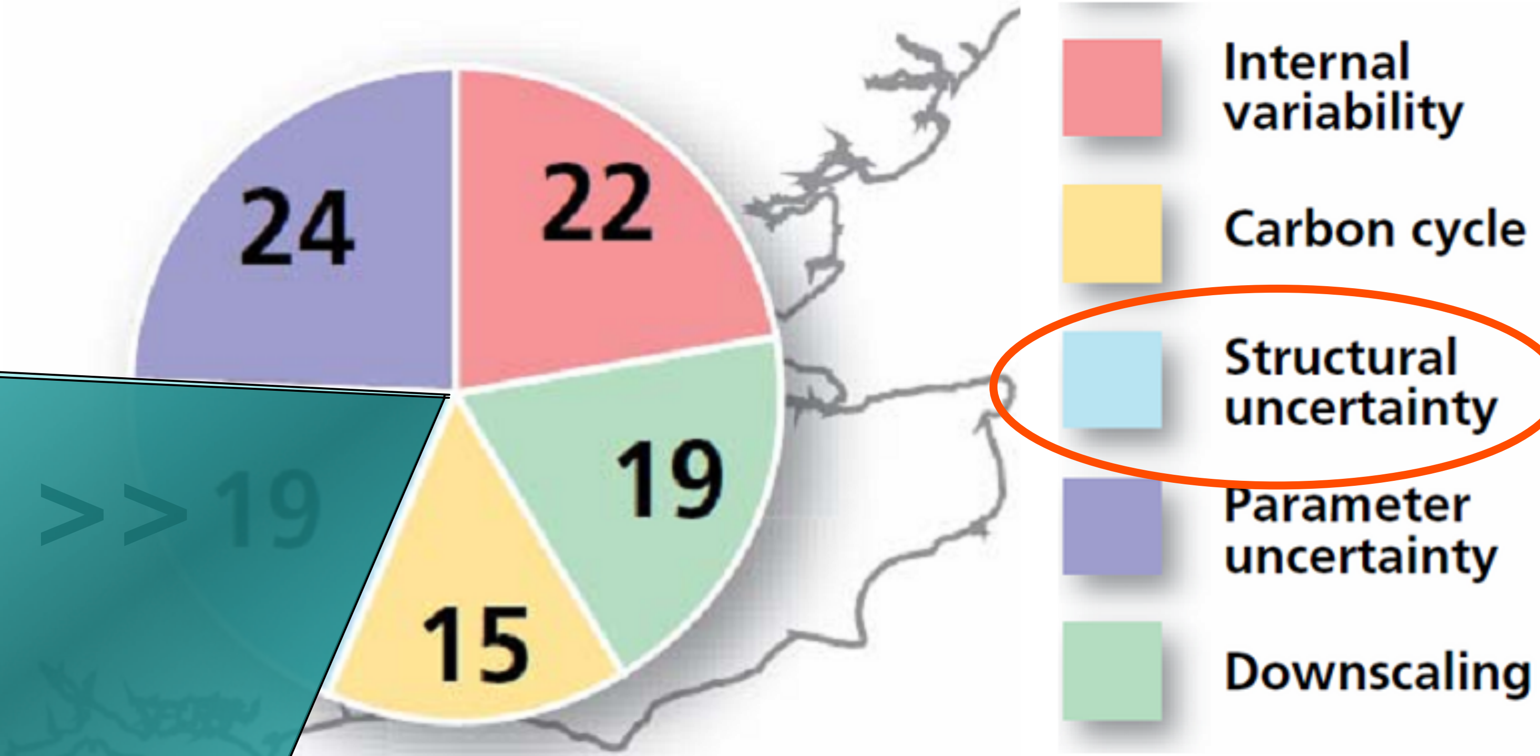


N.B. A **high-variability ridge** lies close to the **diagonal** of the plot (*cf.* also Fraedrich & Böttger, 1978, *JAS*)

* **LFV** ≅ 10–100 days (intraseasonal)

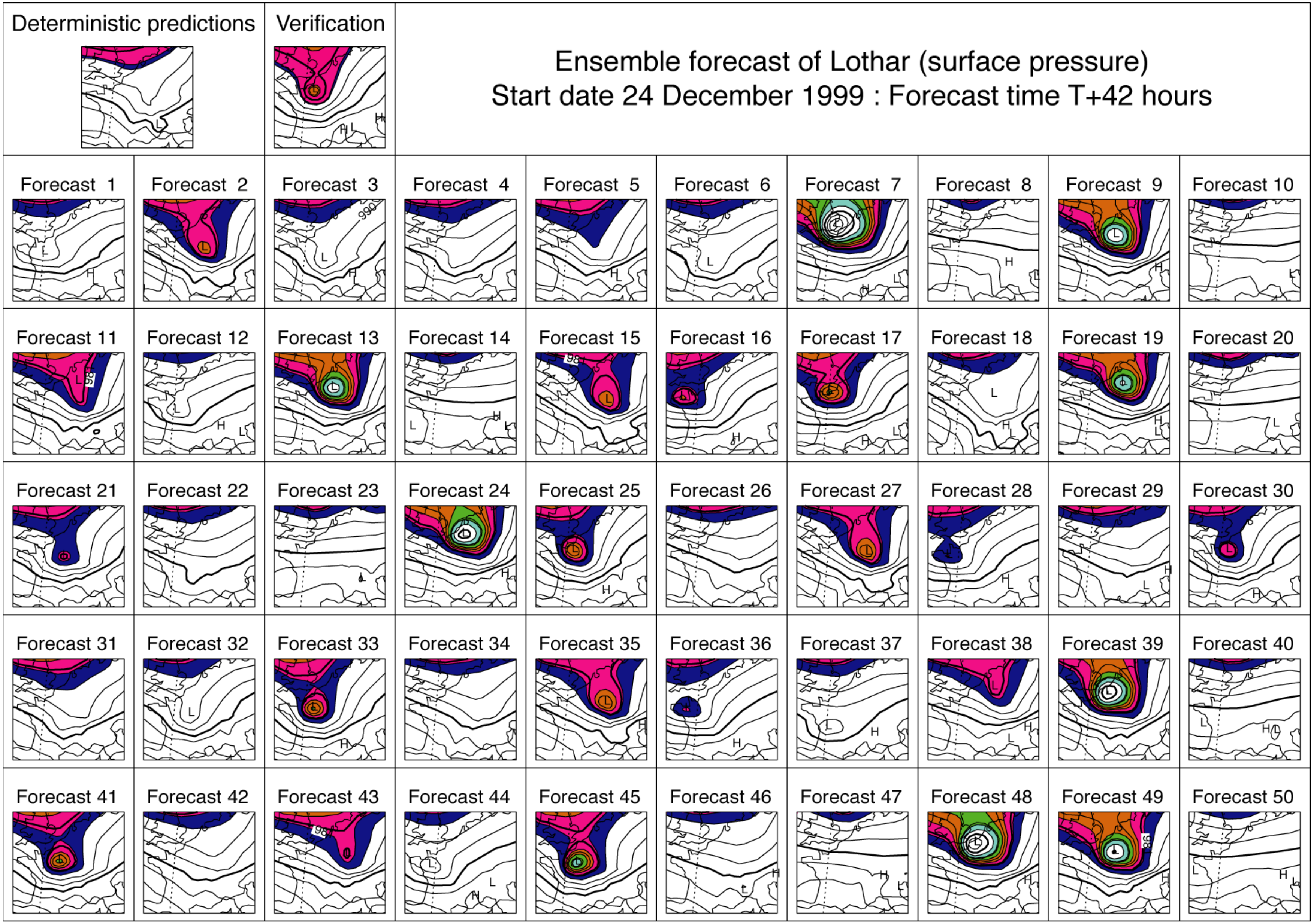
How important are different sources of uncertainty?

- Varies, but typically no single source dominates.



Uncertainties in winter precipitation changes for the 2080s relative to 1961-90, at a 25km box in SE England

Source: Met Office



Courtesy Tim Palmer, 2009

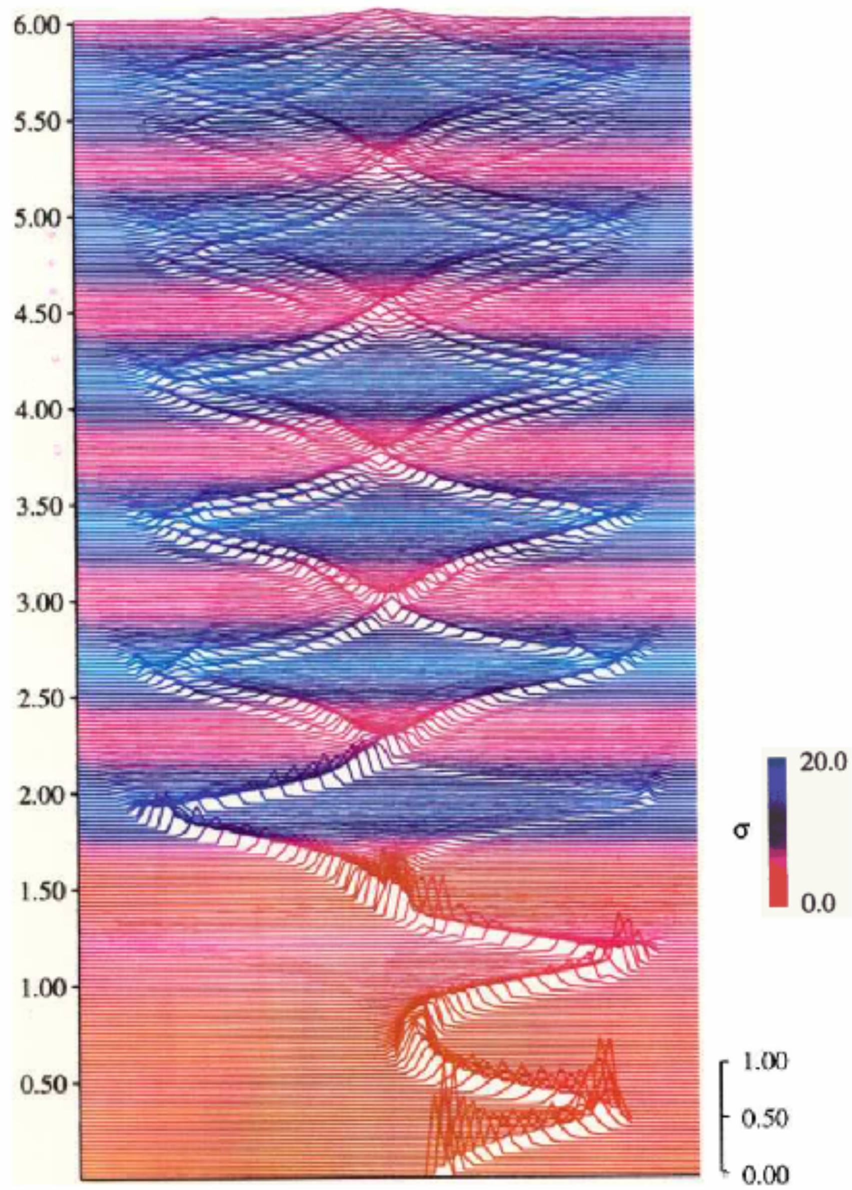
Exponential divergence vs. "coarse graining"

The classical view of dynamical systems theory is:

positive Lyapunov exponent ➔ trajectories diverge exponentially

But the presence of multiple regimes implies a much more structured behavior in phase space

Still, the probability distribution function (pdf), when calculated forward in time, is pretty smeared out



L. A. Smith (*Encycl. Atmos. Sci*., 2003)

Global warming and "global weirding"

"CLIMATE STRANGE
FORGET GLOBAL WARMING—AND
GET READY for GLOBAL WEIRDING
BY BRYAN WALSH"

TIME MAGAZINE, Dec. 29, 2014 – Jan. 5, 2015

"***The New Rule***: For the next few (?) years, global warming will lead to colder, more brutal winters."

- Oh, thank you for the latest prediction from a science journalist — based on interesting but still rather tentative, & hotly debated, suggestions from a few media-loving (& vice-versa) researchers.

- And if this is so certain, why wasn't it predicted by IPCC(*) and other models BEFORE it happened?

(*) Intergovernmental Panel on Climate Change

Transitions Between Blocked and Zonal Flows in a Barotropic Rotating Annulus with Topography

Zonal Flow
13–22 Dec. 1978

Blocked Flow
10–19 Jan. 1963

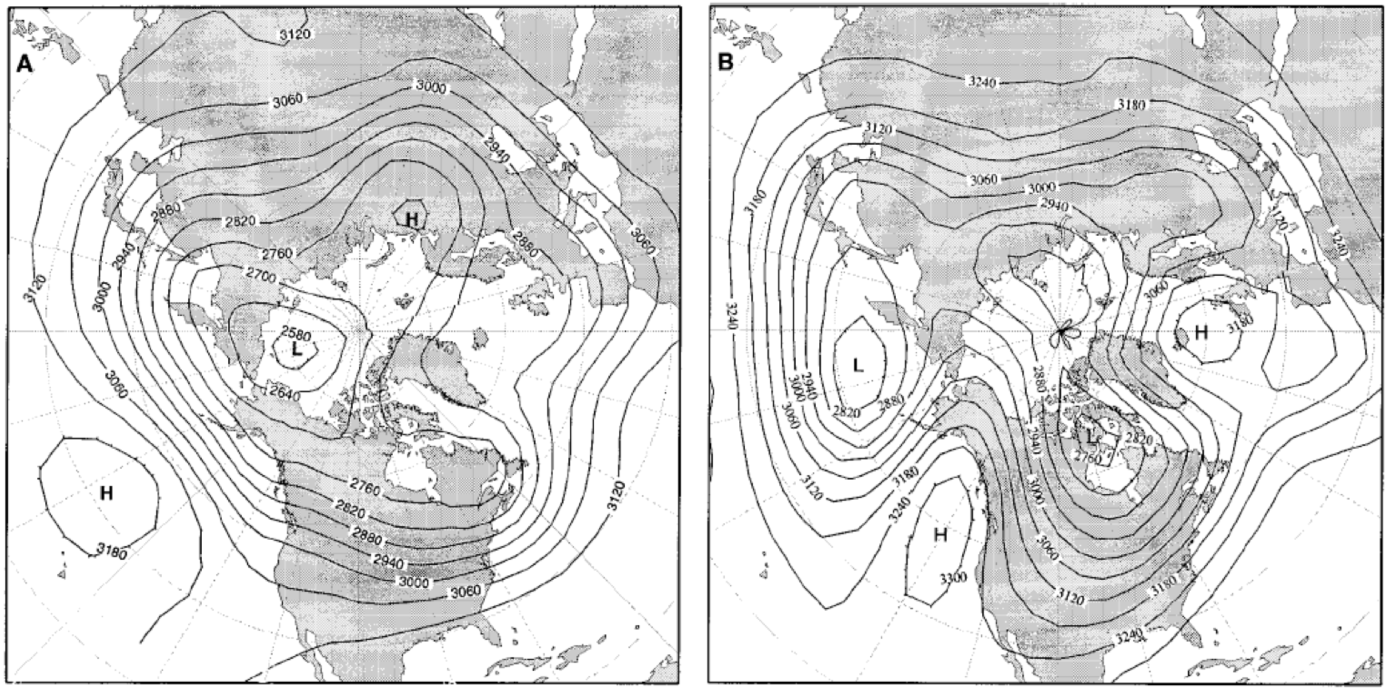
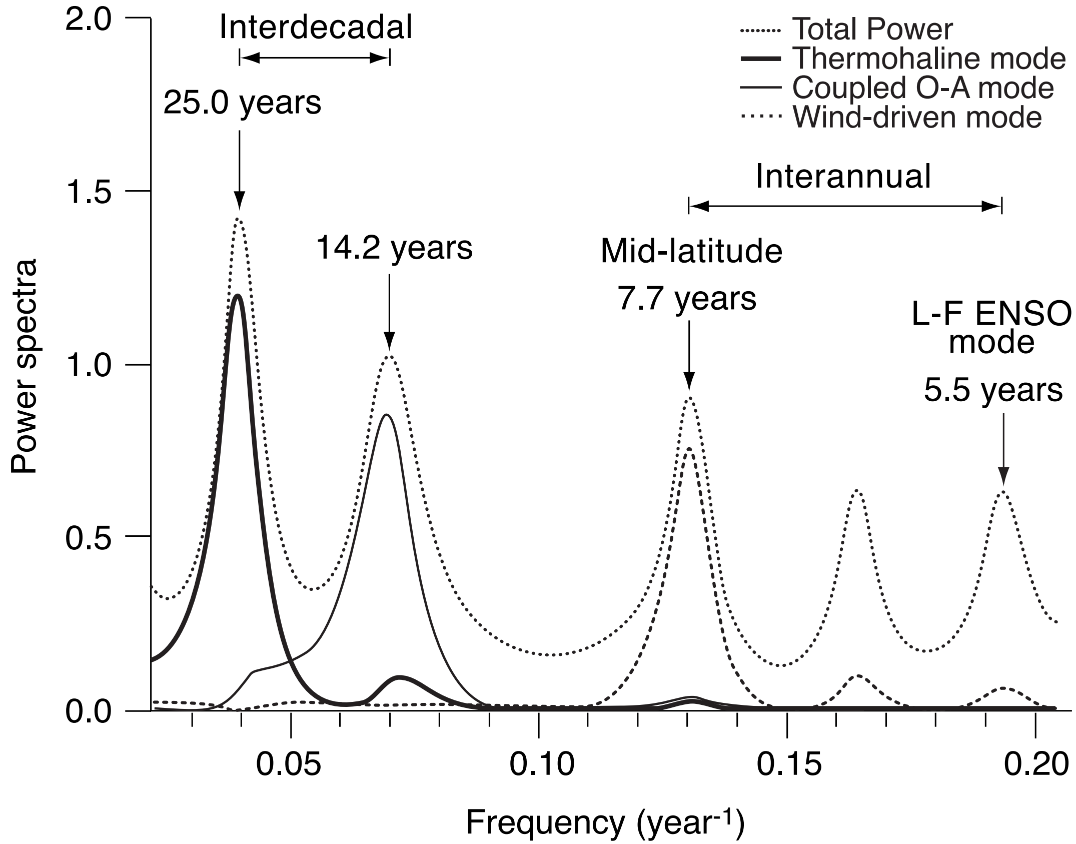


Fig. 1. Atmospheric pictures of (**A**) zonal and (**B**) blocked flow, showing contour plots of the height (m) of the 700-hPa (700 mbar) surface, with a contour interval of 60 m for both panels. The plots were obtained by averaging 10 days of twice-daily data for (A) 13 to 22 December 1978 and (B) 10 to 19 January 1963; the data are from the National Oceanic and Atmospheric Administration's Climate Analysis Center. The nearly zonal flow of (A) includes quasi-stationary, small-amplitude waves (32). Blocked flow advects cold Arctic air southward over eastern North America or Europe, while decreasing precipitation in the continent's western part (26).

Weeks, Tian, Urbach, Ide, Swinney, & Ghil (*Science*, 1997)

SSA (prefilter) + (low-order) MEM

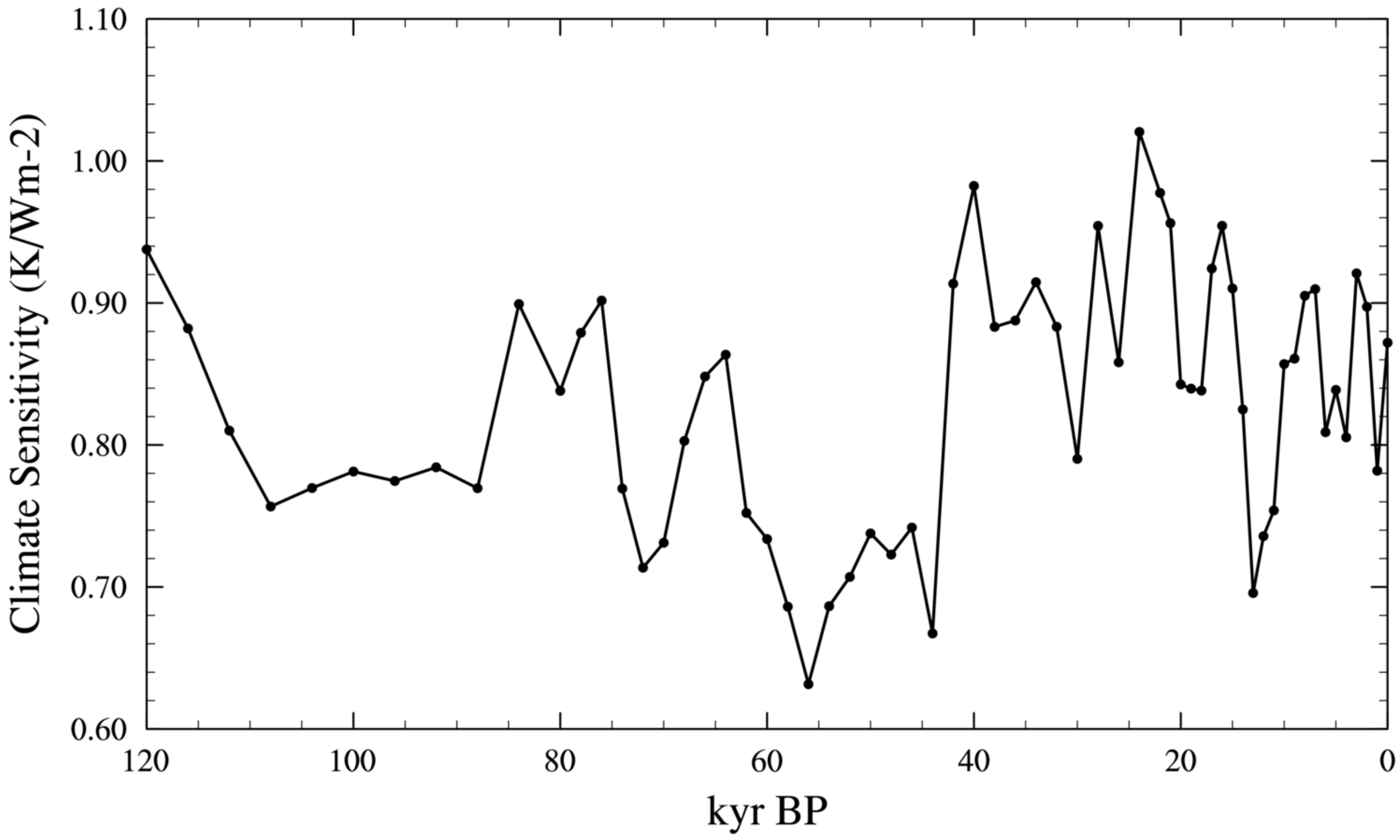
- "Stack" spectrum



In good agreement with MTM peaks of **Ghil & Vautard (1991, *Nature*)** for the Jones *et al.* (1986) temperatures & stack spectra of Vautard *et al.* (1992, *Physica D*) for the IPCC "consensus" record (both global), to wit 26.3, 14.5, 9.6, 7.5 and 5.2 years.

Peaks at 27 & 14 years also in Koch sea-ice index off Iceland (Stocker & Mysak, 1992), etc.
Plaut, Ghil & Vautard (1995, *Science*)

Modeled Climate Sensitivity

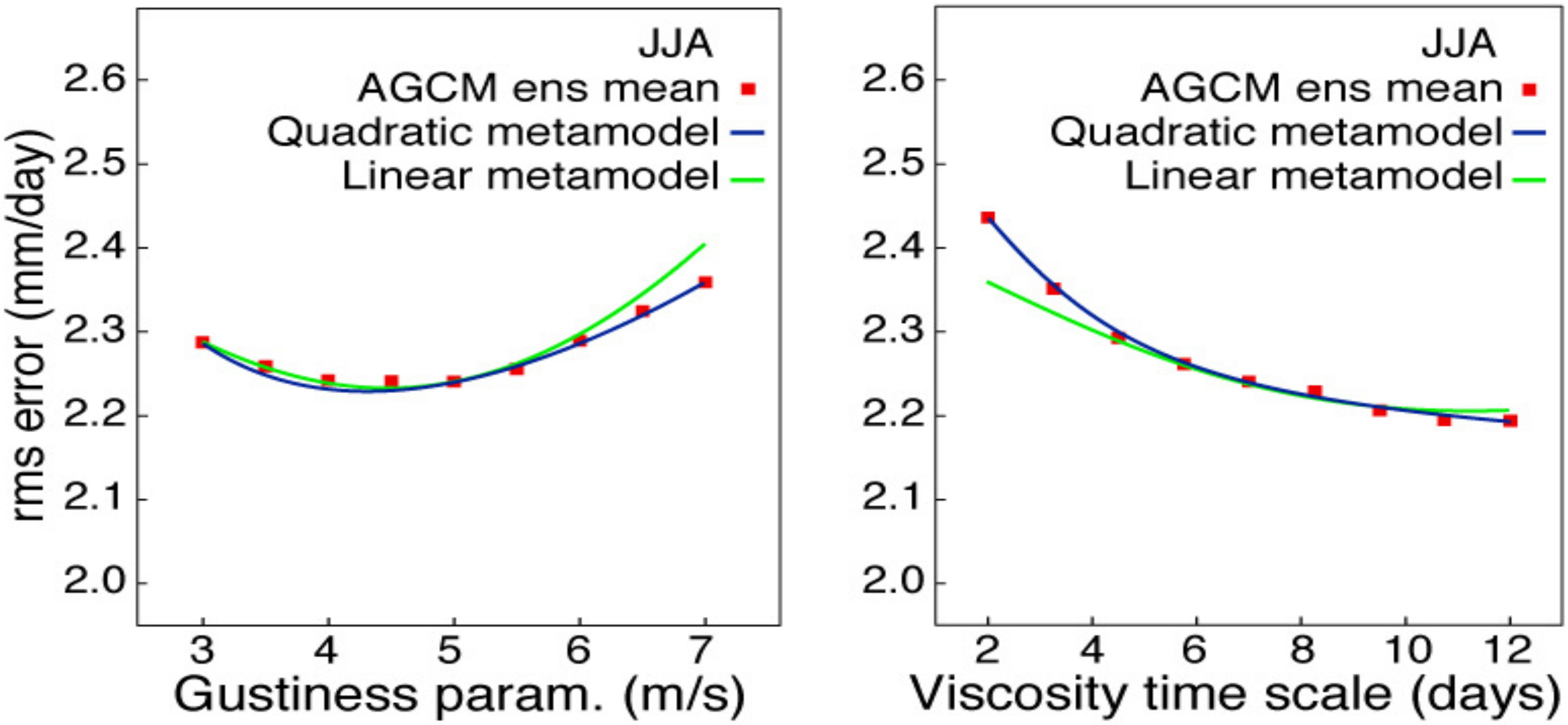


Climate sensitivity as estimated from a series of "snapshot" simulations of paleoclimate using HadCM3.

Courtesy of Paul J. Valdes

Parameter dependence – II

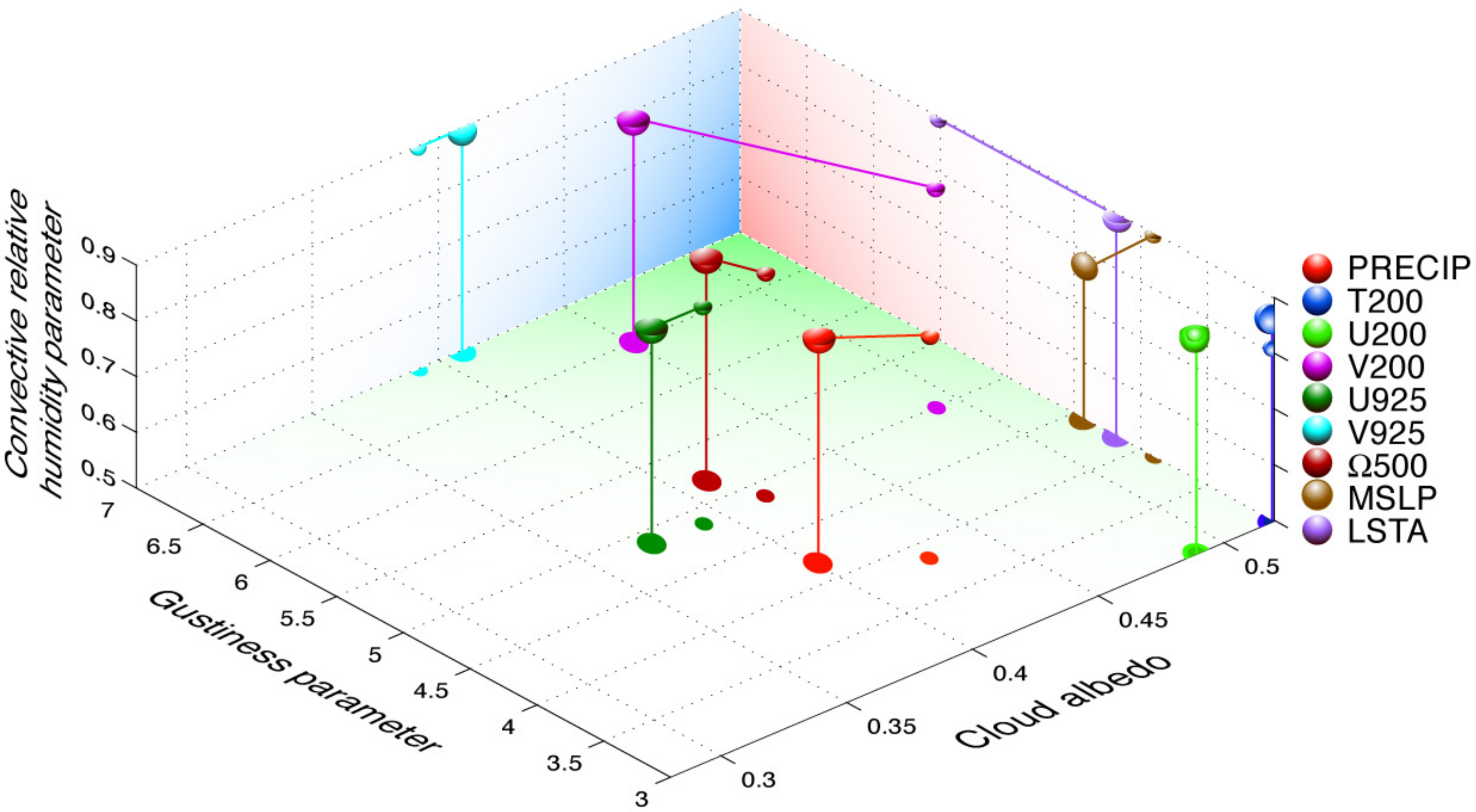
When it is smooth, one can optimize a GCM's single-parameter dependence



ICTP AGCM (Neelin, Bracco, Luo, McWilliams & Meyerson, *PNAS*, 2010)

Parameter dependence – III

Multi-objective algorithms avoid arbitrary weighting of criteria
in a unique cost function:



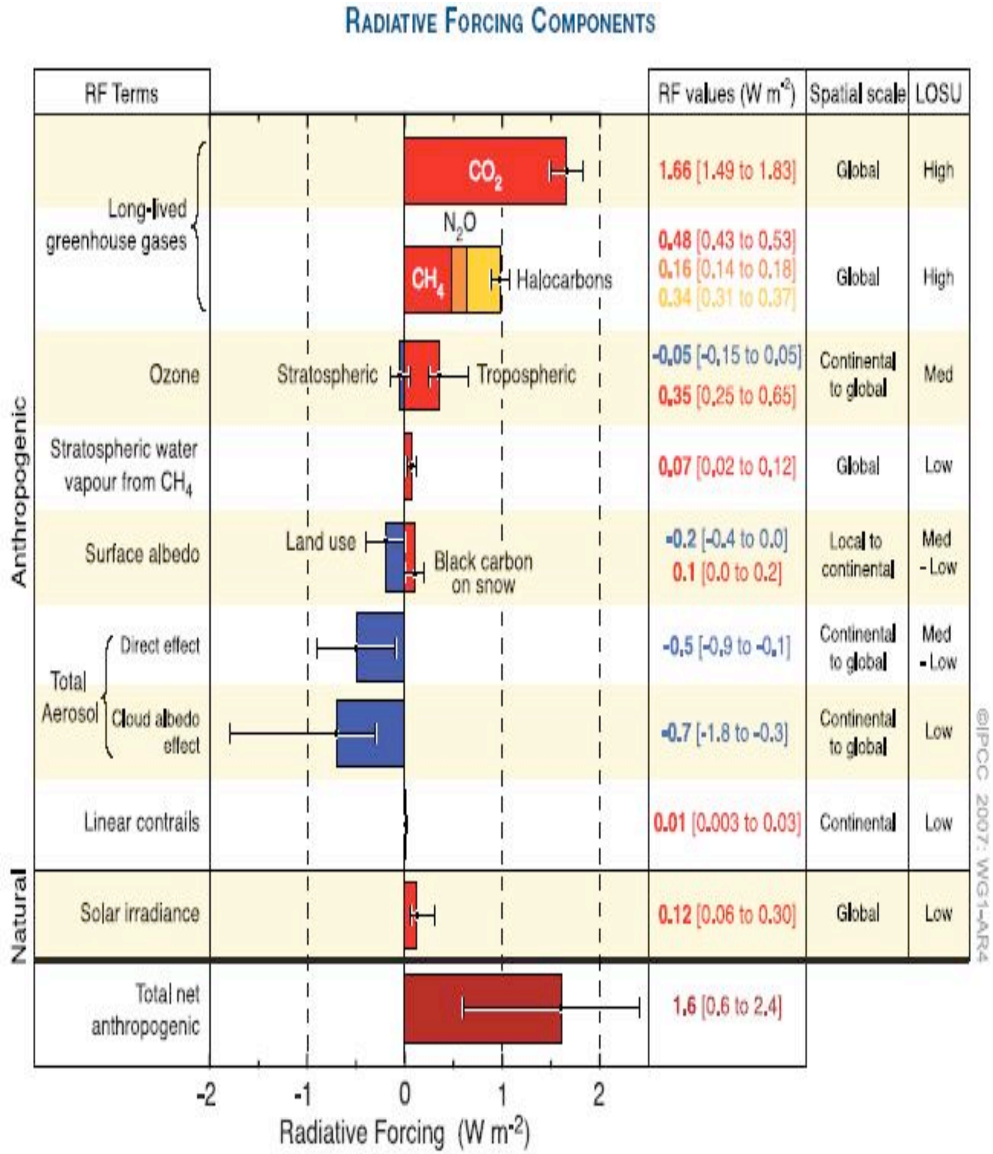
Optimization algorithms that are $\mathcal{O}(N)$ and $\mathcal{O}(N^2)$, rather than $\mathcal{O}(S^N)$, where N is the number of parameters and S is the sampling density.

ICTP AGCM (Neelin, Bracco, Luo, McWilliams & Meyerson, *PNAS*, 2010)

GHGs rise!

It's gotta do with us, at least a bit, ain't it?

But just how much?

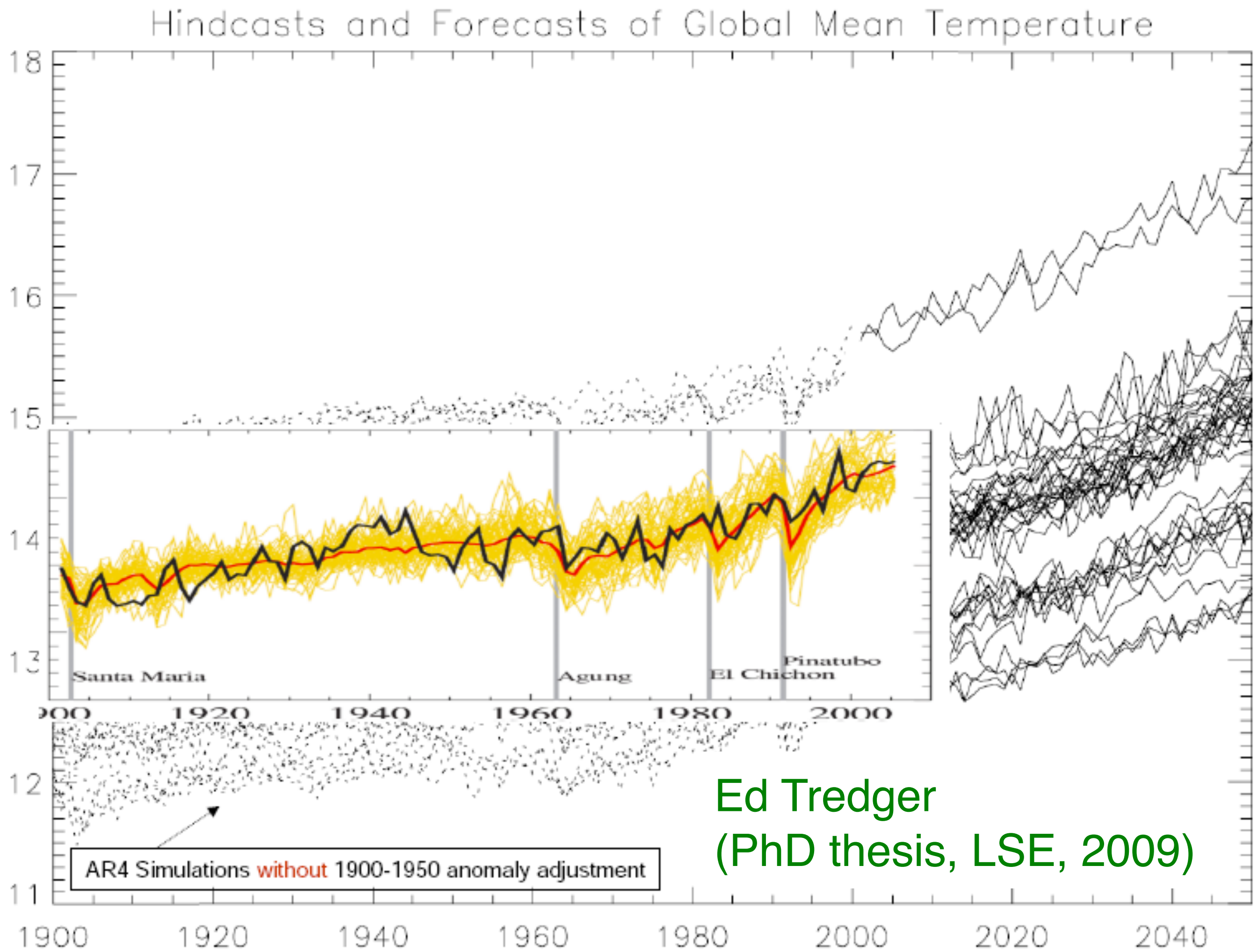


IPCC (2007)

LSE CATS CENTRE FOR THE ANALYSIS OF TIME SERIES

www.lseca

AR4 adjustment of 20th century simulation



Ed Tredger
(PhD thesis, LSE, 2009)

Grantham Research Institute on
Climate Change and
the Environment

L.A. ("Lenny") Smith (2009)
personal communication

Remarks

- We've just shown that:

$$|x(t,s;x_0) - a(t)| \underset{s\to-\infty}{\longrightarrow} 0 \text{ ; for every } t \text{ fixed,}$$

  and for all initial data x_0, with $a(t) = \frac{\sigma}{\alpha}(t - 1/\alpha)$.
- We've just encountered the concept of pullback attraction; here $\{a(t)\}$ is the pullback attractor of the system (1).
- What does it mean physically?
  The pullback attractor provides a way to assess an asymptotic regime at time t — the time at which we observe the system — for a system starting to evolve from the remote past s, $s << t$.
- This asymptotic regime evolves with time: it is a dynamical object.
- Dissipation now leads to a dynamic object rather than to a static one, like the strange attractor of an autonomous system.

Non-autonomous Dynamical Systems - II

Remarks

- We've just shown that:

$$|x(t, s; x_0) - a(t)| \underset{s \to -\infty}{\longrightarrow} 0 \text{ ; for every } t \text{ fixed,}$$

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This theory is the counterpart for randomly forced dynamical systems (RDS) of the *geometric theory* of ordinary differential equations (ODEs). It allows one to treat stochastic differential equations (SDEs) — and more general systems driven by noise — as flows in (phase space)$\times$(probability space).

SDE$\sim$ODE, RDS$\sim$DDS, L. Arnold (1998)$\sim$V.I. Arnol'd (1983).

Setting:

(i) A phase space X. **Example**: $\mathbb{R}^n$.

(ii) A probability space $(\Omega, \mathcal{F}, \mathbb{P})$. **Example**: The Wiener space $\Omega = C_0(\mathbb{R}; \mathbb{R}^n)$ with Wiener measure $\mathbb{P}$.

(iii) A model of the noise $\theta(t) : \Omega \to \Omega$ that preserves the measure $\mathbb{P}$, i.e. $\theta(t)\mathbb{P} = \mathbb{P}$; θ is called the driving system.
Example: $W(t, \theta(s)\omega) = W(t+s, \omega) - W(s, \omega)$;
it starts the noise at s instead of $t = 0$.

(iv) A mapping $\varphi : \mathbb{R} \times \Omega \times X \to X$ with the cocycle property.
Example: The solution operator of an SDE.

Applications to a nonlinear stochastic El Niño model

Chekroun, Simonnet and Ghil, 2011

Timmerman & Jin (*Geophys. Res. Lett.*, 2002) have derived the following low-order, tropical-atmosphere–ocean model. The model has three variables: thermocline depth anomaly h, and SSTs T_1 and T_2 in the western and eastern basin.

$$
\begin{aligned}
\dot{T}_1 &= -\alpha(T_1 - T_r) - \frac{2\varepsilon u}{L}(T_2 - T_1),\\
\dot{T}_2 &= -\alpha(T_2 - T_r) - \frac{w}{H_m}(T_2 - T_{sub}),\\
\dot{h} &= r(-h - bL\tau/2).
\end{aligned}
$$

The related diagnostic equations are:

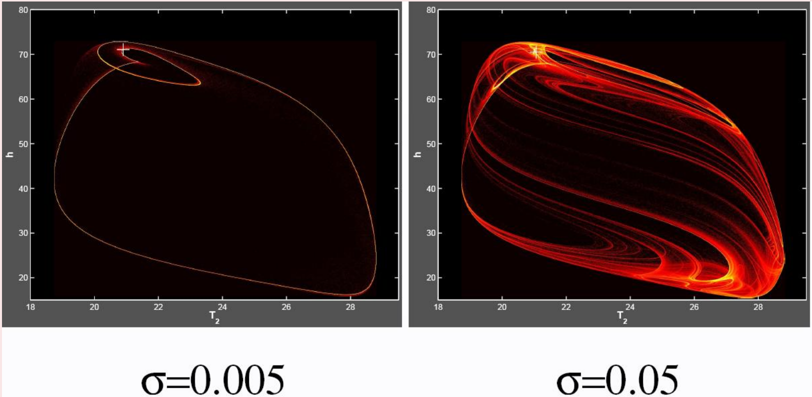
$$
\begin{aligned}
T_{sub} &= T_r - \frac{Tr - T_{r0}}{2}[1 - \tanh(H + h_2 - z_0)/h^*]\\
\tau &= \frac{a}{\beta}(T_1 - T_2)[\xi_t - 1].
\end{aligned}
$$

- τ: the wind stress anomalies, $w = -\beta\tau/H_m$: the equatorial upwelling.
- $u = \beta L\tau/2$: the zonal advection, T_{sub}: the subsurface temperature.

Wind stress bursts are modeled as white noise ξ_t of variance σ, and ε measures the strength of the zonal advection.

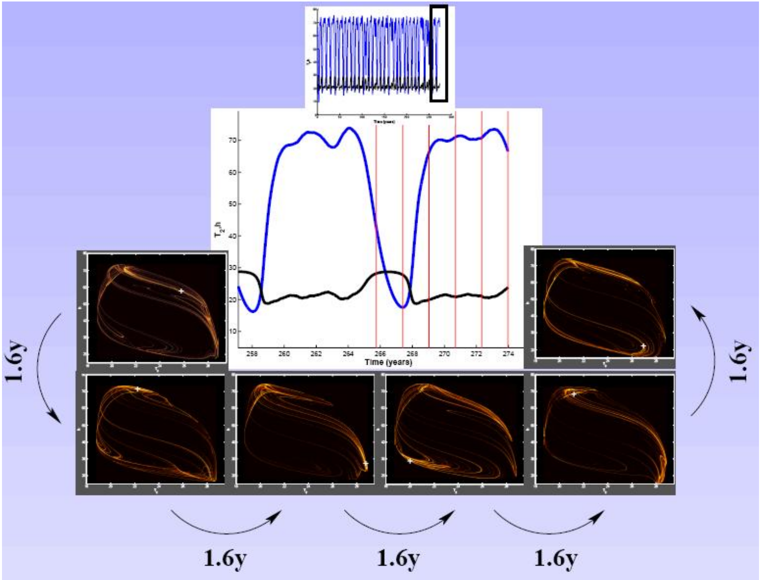
Random attractors: the frozen statistics

Random Shil'nikov horseshoes

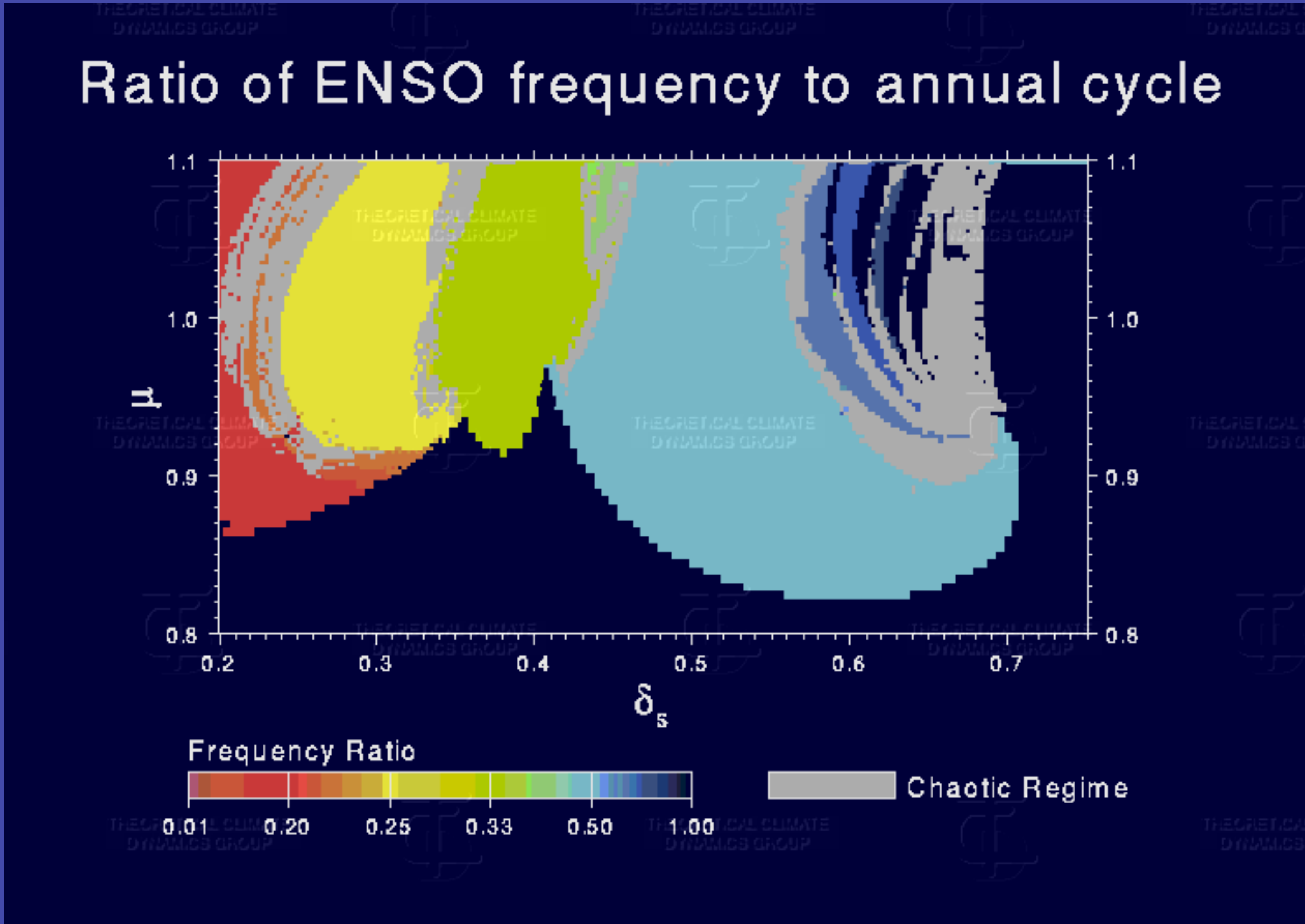


- Horseshoes can be noise-excited,
  left: a weakly-perturbed limit cycle, right: the same with larger noise.
- Golden: most frequently-visited areas; white 'plus' sign: most visited.

An episode in the random's attractor life



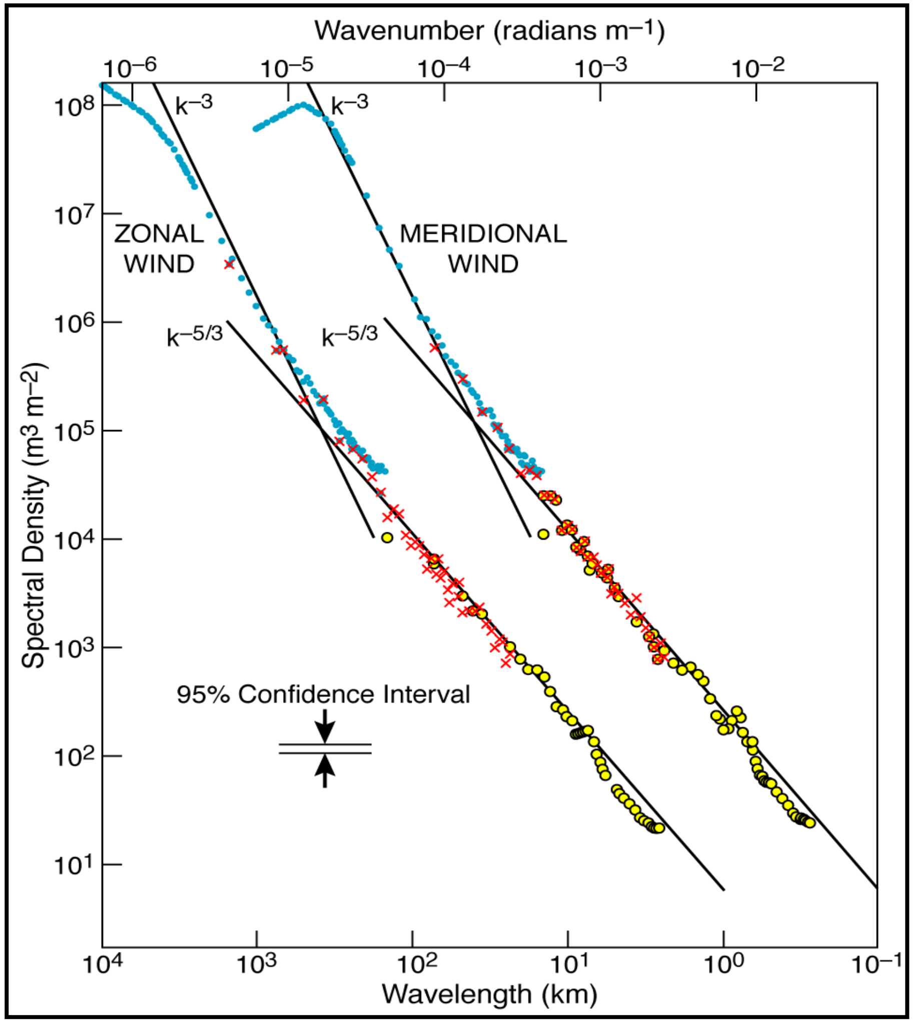
Devil's Bleachers in a 1-D ENSO Model



F.-F. Jin, J.D. Neelin & M. Ghil, *Physica D*, **98**, 442-465, 1996

But deterministic chaos doesn't explain all: there are many other sources of irregularity!

- The energy spectrum of the atmosphere and ocean is "full": all space & time scales are active and they all contribute to forecasting uncertainties.
- Still, one can imagine that the longest & slowest scales contribute most to the longest-term forecasts.
- "One person's signal is another person's noise."



After Nastrom & Gage (*JAS*, 1985)

Climatic uncertainties & moral dilemmas

♥ ***Feed the world today***
or…

♥ … ***keep today's climate for tomorrow?***

Davos, Feb. 2008, photos by *TIME Magazine*, 11 Feb. '08;

see also Hillerbrand & Ghil, *Physica D*, 2008, **237**, 2132–2138, doi:10.1016/j.physd.2008.02.015 .